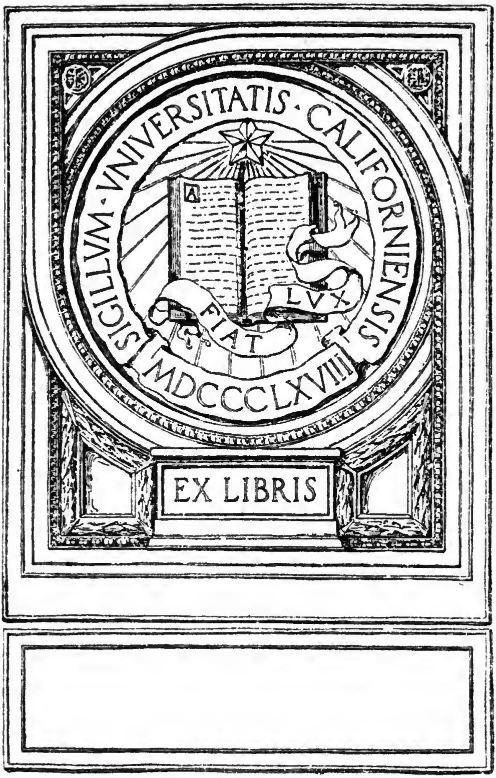
OBSERVATIONAL GEOMETRY

WILLIAM · T · CAMPBELL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PHILLIPS-LOOMIS MATHEMATICAL SERIES

OBSERVATIONAL GEOMETRY

BY

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INSTRUCTOR IN MATHEMATICS IN THE BOSTON LATIN SCHOOL

WITH AN INTRODUCTION BY

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PROFESSOR OF MATHEMATICS AND DEAN OF THE GRADUATE SCHOOL, YALE UNIVERSITY

AND

OVER 300 ILLUSTRATIONS AND DIAGRAMS

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CONTENTS

PART II

POINTS, LINES, ANGLES, POLYGONS, AND CIRCLES. CONSTRUCTIONS, MENSURATION, SIMILAR FIGURES, AND SURVEYING

INTRODUCTION

In the works of nature and of man Geometry plays a most important rôle. The rays of light from the sun suggest the straight line; the surface of still water, the plane; the faces of crystals, a variety of elementary plane figures bounded by straight lines; while the crystals themselves suggest the most common figures bounded by planes. Moreover, the myriad other forms in the animal, the vegetable, and the mineral kingdoms furnish unending variety of symmetrical and complex geometric forms, while the creations of the artist and the architect, and the problems of the engineer and the astronomer, all have their basis in Geometry.

The practice of training pupils early in observing the simple geometric forms and relations of the objects which come under their every-day notice, of teaching them the use of the simplest tools of geometric construction, and of making them familiar with a variety of means of finding lengths, areas, and volumes, is a most natural and potent means of training their powers of observation, and at the same time of cultivating habits of concentrated and continuous attention.

The old arithmetics with their puzzling problems furnished a powerful means for the cultivation of the powers of analysis, but they did not furnish in any adequate sense the careful training of the child's faculties of observation.

It is true that many of their problems were of a practical nature, and were invaluable as a means of familiarizing the student with some of the simple rules of mensuration, and of creating an interest in the methods of making measurements for obtaining the data for problems to which these rules might be applied—problems in finding the contents of bins and boxes, and of calculating the amount of lumber used in their construction; problems in finding the areas of various shaped fields; problems in finding the heights of trees from their shadows, etc.

Such geometric problems often awaken an intense interest, and a desire to know the reasons for the rules employed in their solution, and so create an appetite for the study of Formal Geometry. The want, however, of careful and systematic development of the subject as a means of cultivating the faculties of observation caused a revolt against the arithmetic problems, and resulted in the substitution of nature studies to a considerable extent in the schools for the drill in such problems. But nature studies, which are taught mainly to direct attention to plant and animal life, and to the mere observation of form, fail to give that sharpness to the mental faculties, and that severe training in vigorous thinking, which the consideration of mathematical problems alone can give.

The Observational Geometry combines the training of the nature studies, so far as these educate the eye to keen and intelligent perception, with the training which the more valuable problems of the old arithmetics furnish, and so gives a mental discipline at once rigorous and entirely free from that one-sidedness which either of these systems fosters when taken alone.

It gives the hand dexterity and skill in making drawings and models of geometrical figures. It trains the eye to estimate with accuracy forms and distances. It teaches an ap-

preciation of beautiful and symmetric forms. It seeks out and appropriates methods of accomplishing geometrical results from every source in nature and every employment in life. It is the best stimulant for the inventive faculties. It makes the student familiar with many of the terms and ideas of the physical sciences, and is the open door to the successful study of the formal and the higher branches of Geometry.

ANDREW W. PHILLIPS.

TO THE TEACHER:

The models should be made in the class-room, under the eye of the teacher. The best material is a thin card-board called "light tag stock," which is cheap, and can be procured in quantities cut in sheets of convenient size. The pupil should preserve the completed models in a box, which can serve also as a receptacle for drawing-materials. The question of neatness in workmanship should be settled with the first model.

The number of models to be made will vary, of course, with different classes and individuals, as also will the work which can be left to pupils to do by themselves; but it is intended that instruction shall be largely conversational. The author has treated the cube in greater detail as a suggestion of the method to be employed with other figures.

The pupils should be warned that dimensions given in two systems with the diagrams are alternatives and not exact equivalents.

FOR REFERENCE

MEASURES OF LENGTH, WITH EQUIVALENTS

Metric Table

10 millimetres (mm.)	= 1 centimetre (cm.)	= $\frac{3}{8}$ inches	nearly.
10 centimetres	= 1 decimetre	= $3\frac{15}{16}$ "	"
10 decimetres	= 1 metre	= $39\frac{3}{8}$ "	"
10 metres	= 1 dekametre	= 2 rods	"
10 dekametres	= 1 hektometre	= 20 "	"
10 hektometres	= 1 kilometre	= $\frac{3}{5}$ miles	"
10 kilometres	= 1 myriametre	= $6\frac{1}{5}$ "	"

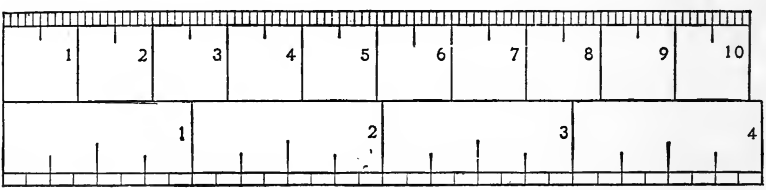
The metre is nearly the ten-millionth part of the distance on the earth's surface from the equator to either pole, first calculated in France, A. D. 1799.

English Table

12 lines	= 1 inch (in.)	= 25 millimetres	nearly.
12 inches	= 1 foot (ft.)	= 3 decimetres	"
3 feet	= 1 yard (yd.)	= 0.9 metres	"
$5\frac{1}{2}$ yards = $16\frac{1}{2}$ feet	= 1 rod (rd.)	= 5 "	"
40 rods = 220 yards	= 1 furlong (fur.)	= 201 "	"
8 furlongs = 5280 feet	= 1 mile (m.)	= 1.6 kilometres	"

The yard is said to have been taken from the length of the arm of Henry I. of England, A. D. 1101.

The upper edge of the following measure is one decimetre long, and is divided into centimetres and millimetres. The lower edge is four inches long, and is divided into quarters and eighths of inches.



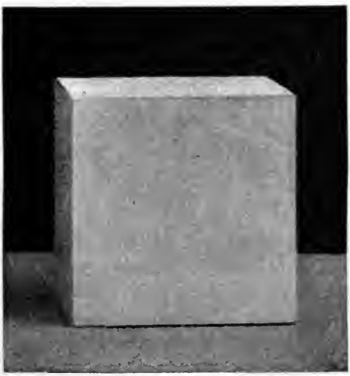
PART I

ELEMENTARY FORMS AND CONSTRUCTION OF MODELS

SIMPLE EXPERIMENTS IN MENSURATION

OBSERVATIONAL GEOMETRY

CHAPTER I

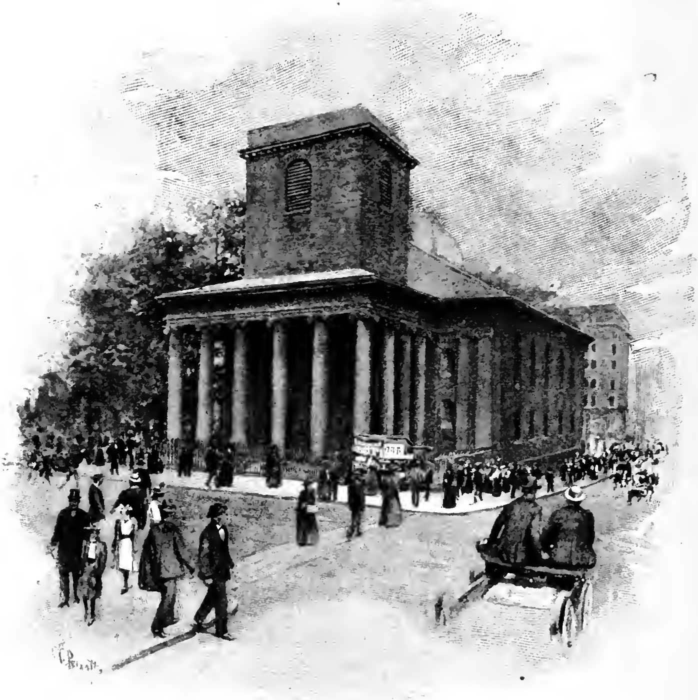


THE CUBE

1. We begin to-day the study of geometry. We are going to model and study some of the principal geometric figures. The picture at the top of this page is that of a *cube;* you have seen things shaped like it, — blocks of stone, glass paper-weights, boxes, and occasionally buildings, or at least some parts of buildings. For instance, the belfry of King's Chapel, from the roof of the portico to the cornice, is a cube.

The sides of a cube are all alike; if you observe the one in the model which is turned toward us, you will see that its

edges are all straight, all of the same length, and where they come together at the corners they meet "straight across" or perpendicularly, so that the corners too are all alike. Now in geometry when anything has four straight edges and four

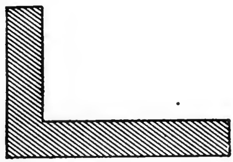


King's Chapel, Boston

corners like this, it is called a *square*. You will remember that we are speaking of only one side of the cube.

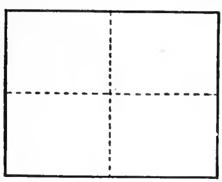
2. **How to draw Right Angles.** When a carpenter wishes to saw a piece of wood straight across, or to make a square corner, he uses that flat piece of steel, which you may have

seen, called the "carpenter's square." As we shall have to draw square corners, we will make something which will serve the same purpose as the carpenter's square.



A Carpenter's Square

Take a piece of rather stiff paper about the size of a sheet of note-paper opened out, fold it once, then turning the paper half around fold it again straight across the first fold, so that the edges of the latter will come just evenly against each other. If you have done this correctly you will find, when you open the paper, that there are two straight creased lines crossing each other straight across or perpendicularly, so that the corners formed by these crossing lines are precisely alike. When straight lines come together as these do, they are said to meet at *right angles*. Now fold the paper up again, twice, as before, and you can use it just as the carpenter uses his square: first, however, beginning at the folded corner, along

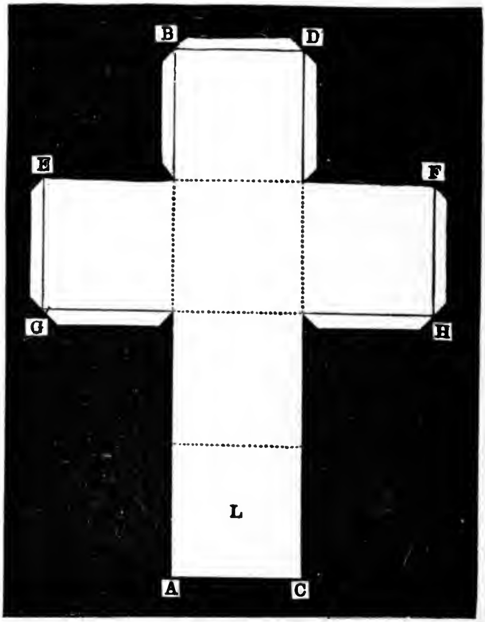


Right Angles

the edge where the paper presents a single fold, make an exact copy of the rule given on the page containing the table of measures of length,—metric or English, whichever one you are to use in constructing the models.

Since we now know something about a square, — and the sides of a cube are all squares, as you will remember, — we can begin to make a model of the cube.

THE CUBE; CONSTRUCTION



3. **How to draw the Diagram for a Cube.** Take a piece of cardboard 2 decimetres 5 millimetres (or $8\frac{1}{4}$ inches) long and 1 decimetre 6 centimetres (or $6\frac{1}{2}$ inches) wide; by means of your folded measuring paper see that the bottom and left-hand edges are straight and perpendicular to each other.

Then, beginning at the bottom of the paper, at A, which is 5 centimetres 5 millimetres (or $2\frac{1}{4}$ inches) from the left edge, draw the straight line AB (see figure) perpendicular to the bottom of the paper, and 2 decimetres (or 8 inches) long. See that the point B is the same distance from the left edge as the point A.

Then beginning again at the bottom of the paper, at *C*, which is 5 centimetres (or 2 inches) from *A*, draw the straight line *CD*, also perpendicular to the bottom of the paper, and of the same length as *AB*. See that *D* is 5 centimetres (or 2 inches) from *B*. Divide the lines *AB* and *CD* each into four equal parts 5 centimetres (or 2 inches) long, and change two parts — the third counting from the bottom — into dotted lines. Draw *BD* and three dotted lines connecting the points of division on *AB* and *CD*. These four lines should be perpendicular to *AB* and *CD*, and should be each 5 centimetres (or 2 inches) long.

We have now four squares, whose sides or edges are all of the same length, and whose corners are all "square" or right angles.

The third square from the bottom has all its sides dotted. The upper side of this square should now be extended to *E* and *F* by straight lines 5 centimetres (or 2 inches) long; the lower side of this same dotted square should be extended in a similar way to *G* and *H;* the point *E* should be connected with the point *G*, and the point *F* with the point *H*. There are now two additional squares, both of which should be tested with the folded paper.

4. **Dotted Lines and Lapels.** In the figure which you now have, the dotted lines are for folding. Lapels for pasting — for which allowance should be made when you cut out the figure — will come on the three free edges of the top square and on the outer and lower edges of the two squares forming the arms of the cross; at first they should be made 5 millimetres (or $\frac{1}{4}$ inch) wide, but after practice they may be made narrower; they are intended to come inside the model.

5. **What is a Diagram?** You have now drawn what is called a *diagram* (di′-a-gram), which means an outline or sketch of something. Your diagram represents the surface of a cube. A diagram may or may not be of the same size as the object it represents; the one you have drawn is of the same size, but if you compare it with the one on page 4, you will see how the two differ, although they represent the same thing. The diagrams in this book are usually smaller than the objects themselves.

6. **How to cut out a Diagram.** Your diagram has the form of a cross. It may now be cut out by trimming closely the border, except where allowance must be made for the lapels. From below cut close up to the lower corners of the interior dotted square, and from the sides cut close in to the two upper corners. Trim off the corners of the lapels.

With the aid of a ruler and the back of a knife-blade or something of the kind, you should crease the dotted and lapel lines; the pencil-marks will thus come inside the figure, which may now be folded and pasted. At first you are likely to use too much paste. If the cardboard is rather

thick, you will do better to cut the folding edges half through with a knife and let the pencil-marks come outside the figure; then probably you would have to use glue instead of paste. You can produce the cleanest, sharpest edges by laying the cardboard on a glass surface and by using a knife instead of scissors, guiding the knife with a ruler. L is the last side to be pasted.

Testing a Plane Surface

7. **1.** How many sides has the cube? Of what shape are they?

2. How many edges?

3. How many corners?

4. Are the sides flat? To test whether a surface is flat, hold against it in several positions something which is known to have a straight edge (like the edge of a ruler), and see if this edge touches the surface all along its length: if it does so in all positions, the surface is flat, and is called a *plane.* The word "plane" is derived from the Latin word *planus,* which means "flat." Plane surfaces of figures are also called *faces,* which is the word we will use hereafter.

5. Have any objects in the room apparently plane surfaces? Perhaps you can test them with a ruler.

6. How many edges bound each face of the cube?

7. Does each edge form a part of the boundary of more than one face? If so, of how many?

8. If you multiply the number of faces by the number of edges which bound each face, by what must you divide the product in order to obtain the number of *different* edges?
9. How many corners has each face of the cube?
10. Does each corner lie in more than one face? If so, in how many?
11. If you multiply the number of faces by the number of corners of each face, by what must you divide the product in order to obtain the number of *different* corners?


Shuswap Lake, British Columbia

A Plane Surface

8. **Horizontal Surfaces.** Observe now the surface of your desk or table, and see if there is any part on which objects would not slide or roll of themselves, however smooth they and the desk might be. If so, that part of the surface of the desk is *horizontal*. The horizon is the line where the sky and the earth's surface seem to come together; and a horizontal plane is one which has the same direction as a plane bounded by the horizon.

The surface of a small body of water at rest is horizontal, such as you see in the picture of the lake. A test whether

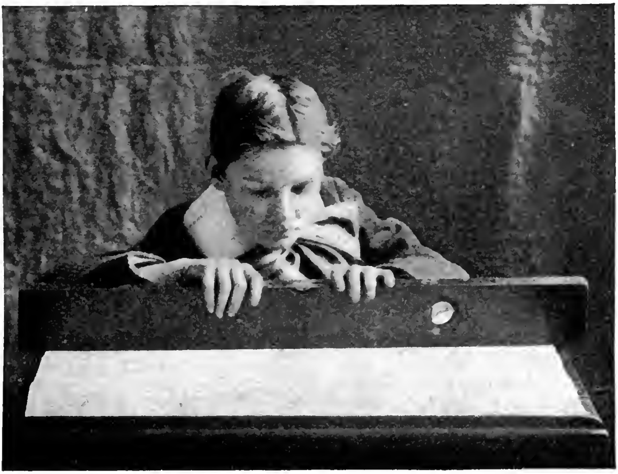
a certain surface is horizontal might be made by seeing if all parts of that surface could touch at the same time the surface of some water at rest.

12. How could you test whether the top of your desk is horizontal, by means of a glass of water?
13. How would you describe floors and ceilings of rooms as commonly laid?
14. Do you know of any floors or ceilings in your school building which are not horizontal?
15. How would you test whether a string drawn tight is horizontal?

9. **Parallel Faces.** Now place the cube on some horizontal part of your desk. The face on which the cube rests is called the *base*. Is the base of the cube horizontal? Is there any other face which is now horizontal? If so, those two faces are parallel (par′-al-lel) to each other. The word parallel is derived from two Greek words meaning "side by side one another." To test whether two faces of an object are parallel, turn the object so that one of the two faces may become horizontal: then if the other face is also horizontal, the two are parallel to each other. Parallel faces cannot meet one another, however far they may be extended. Moreover, parallel faces are at the same distance apart throughout their extents. In the case of the cube the distance between the faces can be measured along the edges. Measure the distance between the two faces you are examining, beginning at each of the four corners of the base. If you find a variation in the four results, either you have made a mistake in measuring, or the cube was not accurately constructed, and is not really a cube.

In practice carpenters lay floors horizontally by the aid of certain instruments, of which a common example is the spirit-level. This consists of a straight bar of wood, in the upper side of which is a glass tube slightly curved and nearly filled with alcohol. When the bottom of the bar is horizontal, a bubble appears exactly in the middle of the tube.

10. **Vertical Planes.** *Vertical* is the opposite to horizontal. It is the direction taken by a plumb line, which is a cord held at one end, hanging motionless, and drawn tight by a weight (usually a piece of pointed lead) attached to the lower end.



Testing a Surface with a Spirit Level

To test whether a plane is vertical, a plumb line is hung near it. If the line can hang freely close to the plane, but without touching it, the plane is vertical.

We will now examine the four side faces of the cube, to compare their directions with that of the base. Placing the cube with its base horizontal, as before, observe that the direction of each of these faces to the base is the same as that of a plumb line to the base. So each of these faces is vertical.

The side faces are also called *perpendicular* to the base. Two planes are perpendicular to each other when they meet

at right angles, so that if the object were turned about and either of the planes were placed in an horizontal position, the other would become vertical.

A Plumb Line and a Vertical Rod

16. Among the four vertical faces of the cube are there any which are perpendicular to each other? Test them by turning the figure over so that one of the two may be horizontal.

17. What is the direction of the ceiling of your room?

18. Of the walls?

19. Of the floor?

20. Is the ceiling parallel to any of these?

21. Is the ceiling perpendicular to any?

22. Are any of the walls parallel to other walls?

23. Are any of the walls perpendicular to other walls?

24. Is the door vertical or horizontal?

25. Does your answer to the previous question depend upon whether the door is open, shut, or ajar?

26. When the door swings on its hinges, does its direction change with reference to the ceiling?

27. With reference to its own wall?

28. With reference to the other walls?

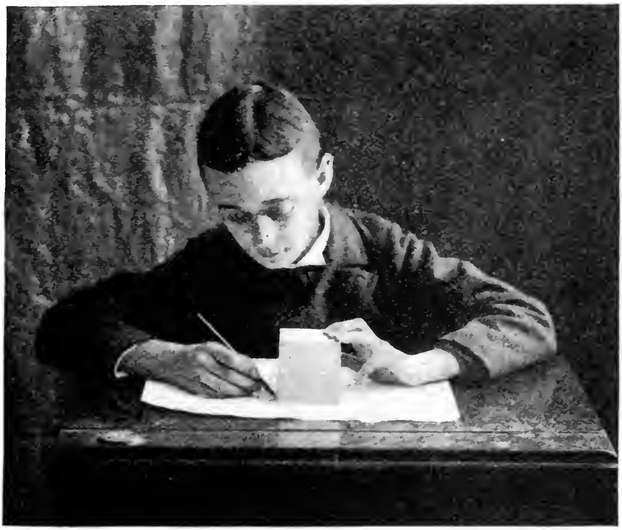
29. Can you hold a book open so that one cover may be perpendicular to the other cover and both may be vertical?

30. Can you, so that one cover may be perpendicular to the other cover, and neither be vertical?

31. Can you do the same and have one cover horizontal? If so, what is the direction of the other cover?

32. How would you distinguish between vertical and perpendicular?

33. Between horizontal and parallel?



Tracing the Base of a Cube

11. **A Test of Geometric Equality.** Next we will examine the shapes and compare the sizes of the six faces. Place the cube on a blank sheet of paper, one face directly before you, and with a pencil trace the outline of the base. Then, without lifting the cube, turn it around so that another face may be before you, and make another tracing of the base in its new position, directly over the first. Turn the cube twice more, and make two more tracings.

With accurate tracings and a true cube the four tracings will look like only one. Likewise, if you turn the cube over upon any other face, you will find that you can trace its outline exactly over the first outline, and in four different positions.

34. How, then, do the six faces compare with each other in shape?
35. How do the six faces compare in size?
36. How many edges bound each face?
37. If you multiply the number of edges of each face by the number of faces, will the product be the number of edges of the cube? Explain your answer.
38. Do the two edges at each corner of each face extend out from each other in the same way?

On the Thames

39. Is it in the same way as a string drawn tight horizontally would extend from a plumb line hanging over one end?
40. Are the edges all of the same length?
41. Are the faces all squares?
42. Describe a cube by its faces, giving their number, shape, and relative size.
43. How many faces of a cube are parallel to any one face?
44. How many faces are perpendicular to any one face?
45. How many edges are parallel to any one edge?
46. How many edges meet any one edge perpendicularly?
47. Can you hold your cube so that eight edges may be horizontal?
48. So that only four edges may be horizontal?
49. So that no edge may be horizontal?
50. So that four edges may be vertical?
51. So that no edge may be vertical?
52. The accompanying picture represents a crew ready to row on the Thames. How many parallel lines do you see?
53. If the crew keep time as they row, will these lines continue to be parallel?

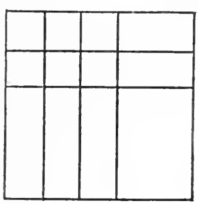
12. **The Three Dimensions in Geometry.** When you measured the distance between the base and upper surface of the cube, you were measuring one *dimension* of the cube, which is called its *thickness*, or *height*, or *depth*.

54. In what instances would you naturally speak of the thickness of objects?
55. Of the height?
56. Of the depth?

Now place the cube as before, horizontally, with one face directly in front of you. You will see that two faces on the sides extend away from you, but are parallel to each other. The distance between these two faces is called the *length* of the cube. Measure the distance in centimetres or in inches.

Lastly, there is a face in the rear, parallel to the one in front; and the distance between these two faces is called the *breadth* or *width* of the cube. Measure the distance in centimetres or in inches.

You have now measured the three dimensions of the cube, — length, breadth, and thickness; and if you have measured correctly, and if the figure was accurately made (that is, if it really is a cube), you have found that all three dimensions of the cube are equal to each other.



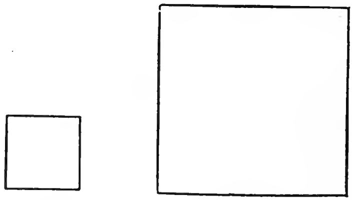
13. **Areas.** We will now examine the size of the faces. Draw on paper a square with edges 5 cm. long. Divide each edge into parts 1 cm. long, and draw lines to connect all the opposite points of division; some of the lines are shown in the annexed figure.

57. What is the shape of the parts into which you have divided your square?
58. Into how many parts have you divided it?

Draw a square with edges 3 cm. long; draw dividing lines as before, and count the number of parts into which the square is divided.

Do the same with a square having an edge of 4 cm.

In these questions you have been finding the *area* of squares. The area of a square means the number of smaller squares into which it can be divided.



Square Cm. Square Inch

If each edge of one of the smaller squares is 1 cm. long, it is called a *square centimetre*, and the larger square is said to contain so many square centimetres.

If each edge of one of the smaller squares is 1 inch long, it is called a *square inch*, and the larger square is said to contain so many square inches.

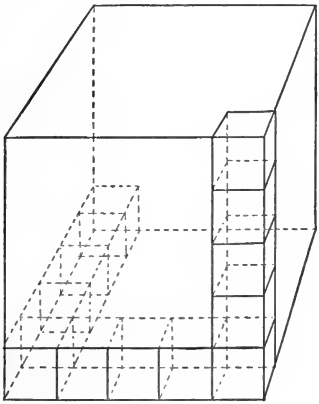
If the square were quite large, like the floor of a room, it would be divided into squares having an edge 1 metre, 1 yard, or 1 foot long, the smaller squares being called square metres, square yards, or square feet.

Can you give a rule for calculating the area of a square without actually dividing the square into smaller ones when you know the length of one edge?

Calculate the area of the entire surface of your cube.

In finding areas you have not considered the question of thickness; for surfaces have only two dimensions, — length and breadth. They have no thickness, being merely the outsides of figures.

14. **Volumes.** Let us now examine the size of the cube. If your cube were solid and yet made of something which could be cut easily, and each edge were divided into five equal parts, the cube could be cut into layers, and each layer could be cut into a number of small cubes.



59. Can you see how many layers there would be?
60. Can you see how many small cubes there would be in each layer?
61. Can you calculate how many small cubes there would be in the whole figure?

Each of these smaller cubes is called a *cubic centimetre*, which means a cube whose edges are one centimetre long. There are a number of cubic centimetres indicated in the figure, but you would not find great difficulty in making yourself a cubic centimetre out of paper, using the diagram at the head of this chapter.

62. How many cubes of the size you have made would it take to form a cube with edges twice as long? Perhaps you can collect the cubes from your classmates, and try the experiment by placing them together.
63. How many cubes would it take to form a cube with an edge three times as long as the first cube?
64. How many cubic centimetres are there in a cube whose edge is 2 cm. long?
65. How many, if the edge is 3 cm. long?

In these questions you have been finding the *volumes* of cubes. The volume of a cube is the number of cubic centimetres, metres, inches, feet, etc., into which it could be divided.

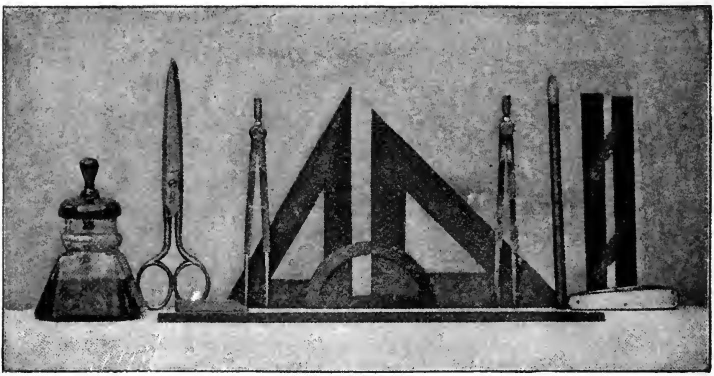
66. Can you give a rule for calculating the volume of a cube when you know its dimensions?

The area of a square is the length of an edge multiplied by itself.

$$Area\ square = s \times s.$$

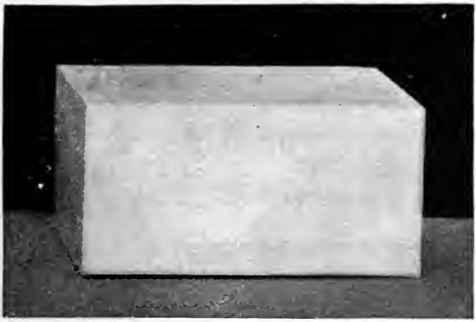
The volume of a cube is the length of an edge multiplied by itself twice.

$$Volume\ cube = s \times s \times s.$$



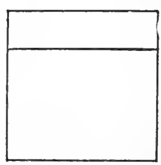
Paste Scissors Dividers Triangles Compasses Pencil Parallel Ruler
Eraser Protractor Knife
Graduated Ruler

CHAPTER II



A PARALLELOPIPED

1. THIS figure is called a *parallelopiped* (par-al-lel-o-pi′-ped), a word which means "having flat, parallel surfaces." The figure has six faces like the cube, which in fact is one kind of parallelopiped; but the name is usually given to figures some at least of whose faces are not squares. If you observe the face which is turned towards us, you will see that like a square it has four edges, meeting one another perpendicularly; but unlike those of the square the edges are of two different lengths, the opposite ones being equal. Such a face is called a *rectangle*, which means "having right angles."



Draw a square with an edge of any length, say 5 cm. (or 2 in.) and cut it out from the paper. With the aid of your folded measuring paper, rule a line straight across, perpendicular to the edges it meets. Then cut

Faneuil Hall, Boston

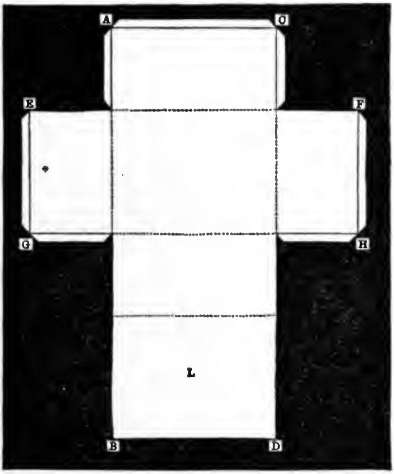
the square into two parts along the line you have just drawn. Each of these parts will be a rectangle.

You will notice that the opposite edges of each rectangle are parallel; and if you fold the rectangle over so that the opposite edges may come together, you will see that they are equal.

Also you can cut each rectangle into smaller rectangles, being careful to make the dividing lines perpendicular to the edges they meet; and you can turn a rectangle into a square by cutting off the right amount.

The parallelopiped is by far the commonest of all forms in architecture, occurring repeatedly in the parts of buildings. The picture of Faneuil Hall, for example, shows five distinct parallelopipeds, — three in the body of the building, one in the chimney, and one in the base of the cupola. All the faces, except two in the cupola, are rectangles; so we have here five rectangular parallelopipeds.

We will now make a model of the parallelopiped.



2. The diagram will need paper 25 cm. 5 mm. × 21 cm. (or $10\frac{1}{4} \times 8\frac{1}{2}$ in.). *AB* and *CD* are each 2 dcm. 5 cm. (or 10 in.) long and 1 dcm. (or 4 in.) apart; that is, *AC* and *BD* are each 1 dcm. (or 4 in.) long.

AB and *CD* are each divided into parts beginning at *A* and *C* as follows: 5 cm. (or 2 in.), 7 cm. 5 mm. (or 3 in.), 5 cm. (or 2 in.), and 7 cm. 5 mm. (or 3 in.).

EF and *GH* are each 2 dcm. (or 8 in.) long, extending at each end 5 cm. (or 2 in.) beyond *AB* and *CD*, which they cross at the first and second points of division from *A* and *C*. *EG* and *FH* are each 7 cm. 5 mm. (or 3 in.) long.

Wherever the lines cross they are perpendicular to each other.

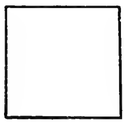
3. **1.** How many faces has this figure?

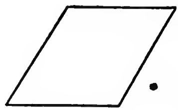
2. How many edges?

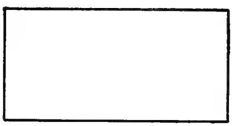
3. How many corners?

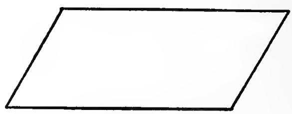
4. If you place the figure with any face horizontal, will any other faces also be horizontal? If so, how many?

5. What other name could be given to those faces, comparing their directions with each other?
6. If the base were horizontal, would any faces be vertical? If so, how many? What other name could be given to those faces, comparing their directions with that of the base?
7. Is it true that each face of this figure is bounded by two pairs of parallel edges?
8. Is it true that the edges of each face which meet are perpendicular to each other?
9. How would you answer the last two questions in the case of the faces of the cube?
10. Are the faces of the new figure squares? If not, what difference do you see between them and squares?


Square


Rhombus


Rectangle


Rhomboid

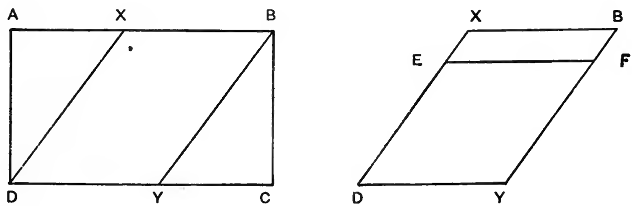
4. **Quadrilaterals. Parallelograms.** Any face which is bounded by four edges is called a *quadrilateral* (quad-ri-lat′-er-al), which means "four-sided." The square and the rectangle, however, belong to a particular class of quadrilaterals, called *parallelograms* (par-al-lel′-o-grams), which means "parallel marks or lines." You will remember that the angles of the square and rectangle are right angles: now, if in these figures you change the direction of two opposite sides as regards the other two, there will no longer be any right angles, but there will be two acute and two obtuse angles in each figure. This is what is called "distorting" a figure. If you distort the square and the rectangle you

form two other parallelograms: from the square you form the *rhombus* (rhom′-bus), and from the rectangle you form what was formerly called the *rhomboid* (rhom′-boid), though this term is no longer in general use, and the figure is commonly known simply as a parallelogram. When no variety is specified, "parallelogram" is understood to mean the rhomboid.

A *rhombus* is a parallelogram whose edges are all equal, but whose angles are not right angles.

The word means "something which can be whirled around," the shape having some likeness to an old-fashioned whirling spindle.

The rhomboid and rhombus can be formed from the rectangle by cutting.



Draw a rectangle *ABCD* with edges 7 cm. and 4 cm. (or $3\frac{1}{2}$ in. and 2 in.) and cut it out from the paper.

Beginning at two opposite corners, *A* and *C*, measure on opposite edges the equal lengths *AX* and *CY* 3 cm. (or $1\frac{1}{2}$ in.); and draw the lines *DX* and *BY*. Then cut through the lines *DX* and *BY*. The part left, *DYBX*, is a rhomboid. You will see that the opposite edges are parallel; and by measuring the lengths you will find that the opposite edges are also equal. If you do the work accurately, the lengths will prove to be 4 cm. and 5 cm. (or 2 in. and $2\frac{1}{2}$ in.).

Then, beginning at the ends of one of the shorter edges, measure on the longer edges *XE* and *BF* each 1 cm. (or $\frac{1}{2}$ in.) long; draw the line *EF*; and cutting along this line divide the rhomboid into two parts. The smaller of the two parts will be another rhomboid. The greater part, *EFYD*, will be a rhombus.

All four forms of the parallelogram agree in having the opposite edges parallel; and all have the opposite edges equal.

In what particular respect does a rectangle resemble a square?

In what particular respect does a rhombus resemble a square?

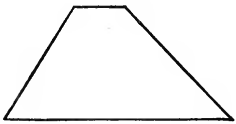
In what does a rectangle differ from a square?

In what does a rhombus differ from a square?

Take a piece of string, tie three knots in it, and lay it down on your desk in the form of a square, the knots coming at the corners. Then change the square into a rhombus having the same knots at the corners.

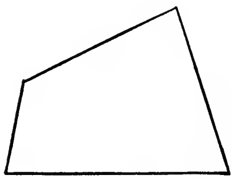
Lay the same string down in the form of a rectangle with knots at the corners. Will the knots be the same as you used for the square?

Can you change the rectangle into a parallelogram without making new knots?



Trapezoid

There are two more forms of plane surfaces bounded by four edges, — the *trapezoid* (trap′-e-zoid) and *trapezium* (trap-e′-zi-um).



Trapezium

A *trapezoid* has two edges which are parallel and two which are not parallel.

The word means "like a table."

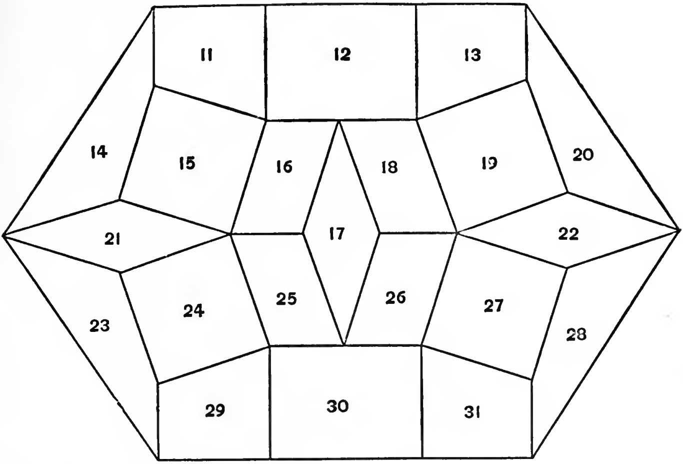
A *trapezium* has none of its edges parallel.

The word means "a little table."

Do you see how to change a parallelogram into a trapezoid by cutting it once?

How many times must you cut to change a parallelogram into a trapezium?

In the following collection of quadrilaterals, judging by the eye, give the name of each: —



32. When you measured the cube, what did you find about the three dimensions?
33. Are the three dimensions of your rectangular parallelopiped equal to each other?
34. What is the length, — that is, the longest dimension?
35. What is the breadth?
36. What is the thickness?
37. Can you place the figure so that its thickness or height may be its greatest dimension and its length may be its smallest dimension?
38. What are the dimensions of the two greatest faces of this figure?
39. Of the next greatest?
40. Of the smallest?
41. How are those faces which are equal situated with respect to each other?

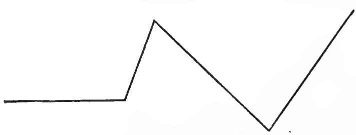
5. **Lines.** We will now examine the edges more particularly. Edges are *lines;* furthermore, they are the only true "lines" in the geometric sense of the word. A "line" in geometry has only one dimension, — length; it has no breadth or thickness. You can *represent* a line, however, by making a mark on a surface with a pen, a pencil, or even a crayon. The boundaries of surfaces are lines: wherever two surfaces meet each other, there is a line in common.

A *straight* line is formed where two plane surfaces meet. Thus the edges of cubes and parallelopipeds are all straight

lines. The word straight originally meant "stretched," a cord drawn tight representing a geometric straight line. Notice that a straight line keeps the same direction through-

A Straight Line

out its length, and that straight lines are all of one kind. though we sometimes speak of a "broken" or a "zigzag" line, which is really a collection of straight lines.



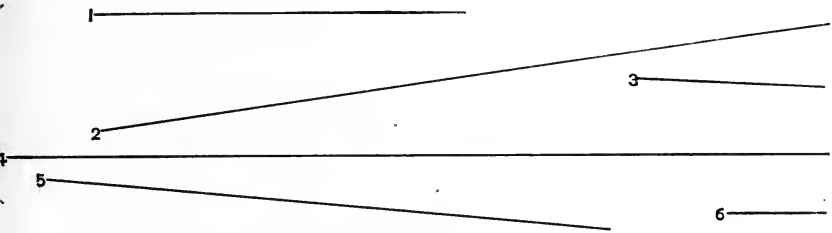
A Broken Line

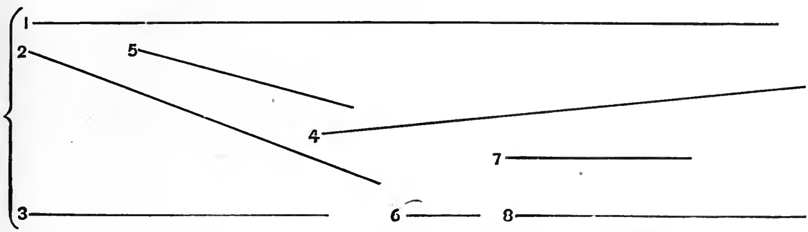
6. **The length of a straight line** is measured by applying to it any unit which may be agreed upon as convenient, such as a centimetre, metre, kilometre, inch, foot, mile. For short lines an inch or a centimetre is a convenient unit; for long lines, a mile or a kilometre.

Two sets of measurements are in common use, the metric and the English. Both have already been given.

The English system, being in common use in the United States, is likely to seem easier; but with practice in actual measurements, the metric system will be found the simpler to use. You should, however, accustom yourself to make measurements in each system, first judging by the eye, and then measuring accurately with the yard or metre stick.

42. Judge by the eye the lengths of the following lines, and then test them by accurate measurements: —

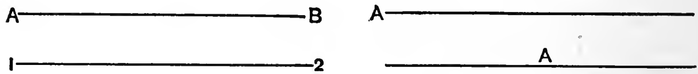




A straight line is the shortest which can be drawn between two of its parts, — for instance, its ends. This affords one test whether a given line is straight or not; for if the shortest distance between the ends is equal to the length of the line, then the line is straight. Let some boy hold a string against the blackboard by the ends so that it may sag a little. Let another boy measure the distance between the ends with a ruler (the edge of which is supposed to be straight), and compare the result with the length of the string.

From the beginning you have been measuring the edges of figures as if you knew them to be straight lines. This was

right; for plane surfaces always form straight lines where they meet or cut each other.



Lines are commonly designated by two letters or two numbers, one being placed at each end. But a line is sometimes designated by a single letter or number placed anywhere upon it.

7. Suppose you wish to draw a straight line of a required length.

If the length of the line you are to draw is given in decimetres or in inches, you can draw the line with the aid of a ruler having a graduated edge, such as is represented on the page containing the table of measures of length. It is a problem you have done since you began this book.

If the length of the required line is not given in numbers, but is shown by another line whose length in numbers is not known, you can perform the problem in either of two ways.

Suppose you are required to draw a line equal in length to AB.

First, you may measure the length of AB with a graduated ruler, and then draw another of the same length. If you find AB to be 3 cm. long, you have only to draw another line 3 cm. long, and the problem will be performed. This way is called "doing the problem by arithmetic." The difficulty is that the length of AB may not correspond exactly to any distance shown by your graduated scale. The next method is preferable.

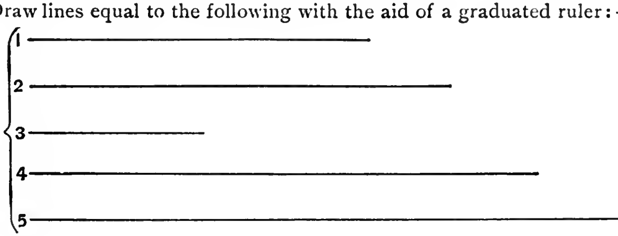
Secondly, you need not find the length of AB in numbers, but instead you may mark on a strip of paper which has a smooth edge two dots showing the length of AB; and then

having drawn a line of any length, you can mark off on it the distance shown by the two dots.

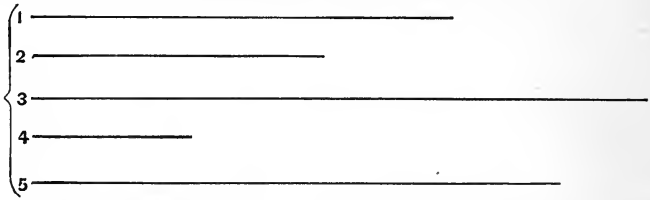
Measuring a line with a pair of dividers

There is also an instrument which is used for this purpose, called "dividers," or "a pair of dividers." This has two prongs opening on a hinge, so that the distance between the pointed ends shows the length of a line. This method is called "doing the problem by geometry."

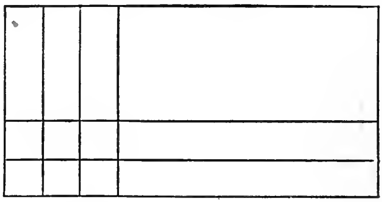
43. Draw lines equal to the following with the aid of a graduated ruler: —



44. Draw lines equal to the following "by Geometry": —



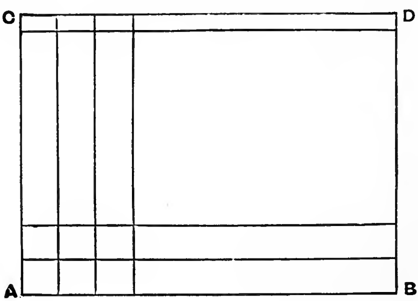
8. **Area of a Rectangle.** Draw on paper a rectangle 10 cm. long and 5 cm. wide, representing one of the faces of your



parallelopiped. Divide the edges into parts 1 cm. long, and draw lines connecting the opposite points of division, as indicated in the diagram.

45. What is the shape of the parts into which you have divided the rectangle?
46. Count the number of these parts.
47. Can you say that there are ten rows with five in each row?
48. Is it also true that there are five rows with ten in each row?
49. What would you say is the area of this rectangle?

Next draw on paper a rectangle 10 cm. long and 7 cm. 5 mm. wide, representing another face of your parallelopiped.



Divide the two longer edges, *AB* and *CD*, into parts 1 cm. long. Then, beginning at *A* and *B*, mark off on *AC* and *BD* parts 1 cm. long as far as they will go. Draw lines as before, connecting the opposite points of division.

50. Count the number of whole squares thus formed.
51. Count the number of the parts which remain.
52. How many of those parts would it take to make one of the squares?
53. How many squares would those parts make if cut out and matched together?
54. Can you say that there are ten rows with seven and one-half squares in each row?
55. What would you say is the area of this rectangle?

Lastly, draw a rectangle seven and one-half centimetres long and five centimetres wide; divide it into squares and parts of squares as before.

56. Count the number of squares.
57. Count the number of other parts.
58. What would you say is the area of this rectangle?
59. Can you give a rule for calculating the area of a rectangle when you know its length and width?
60. Calculate the area of the entire surface of your parallelopiped.

9. **Volume of a Parallelopiped.** The volume of a parallelopiped is found by the method you used with the cube, the

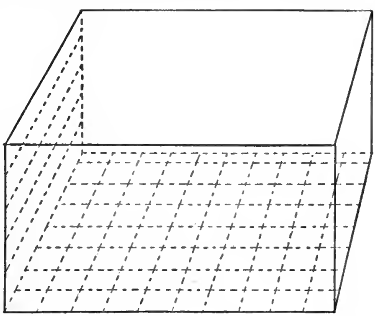


figure being divided into small cubes. The height shows the number of layers of cubes, and the area of the base shows the number of cubes in each layer.

61. The base of your parallelopiped is 10 cm. long and $7\frac{1}{2}$ cm. wide. How many square centimetres are there in its area?

62. How many cubic centimetres, therefore, are there in one layer?
63. The height is 5 cm. How many layers of cubes, therefore, are there?
64. What is the total number of cubic centimetres in the volume of the figure?
65. Can you give a rule for calculating the volume of a parallelopiped when you know the three dimensions?

10. **A Practical Experiment.** You will find it interesting to try practical experiments in comparing the volumes of figures which you make. Now, as the edges of your cube are 5 cm. long, its volume is 125 cubic centimetres. As the dimensions of your parallelopiped are 10, $7\frac{1}{2}$, and 5 cm., its volume is 375 cubic centimetres, which is exactly three times the volume of the cube. The parallelopiped, therefore, ought to hold three times as much as the cube; and you can test this by filling the cube with sand, sawdust, or water, etc., and pouring the contents into the parallelopiped until the latter is full. For this it will be better to have special figures with one side open. If you give the figures a coat of thick varnish, inside and out, they will hold water.

When you are done, preserve the two figures carefully; for we shall need them in future experimenting.

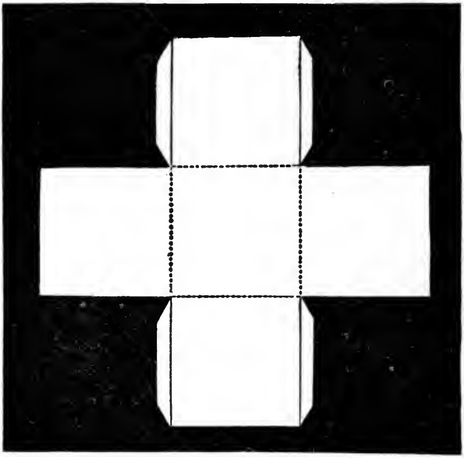


Diagram for a Measuring Cube

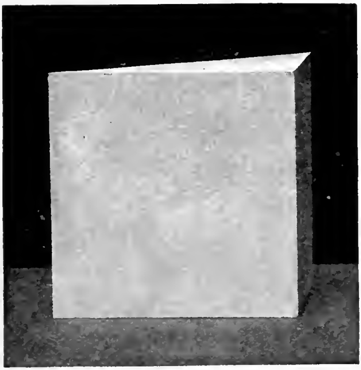
The area of a rectangle is the product of its two dimensions.

$$\textit{Area rectangle} = a \times b.$$

The volume of a parallelopiped is the product of its three dimensions.

$$\textit{Volume parallelopiped} = a \times b \times c.$$

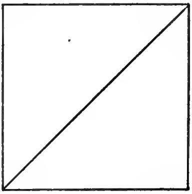
CHAPTER III



A PRISM

1. THIS figure is called a *prism*, which means "something sawed off;" that is, a prism is a part of another figure; when you have made the prism you will see what is the figure of which it is a part. The face turned towards us is a square; the one which extends to the rear on the right is another square; the one towards the left is a rectangle.

Each of the faces at the top and bottom is a *triangle* (tri′-an-gle), which means "three-cornered."



Draw a square with an edge of 5 cm. (or 2 in.), and cut it out from the paper; draw a line from corner to corner, and then cut the square into

two parts along this line: each part will be a triangle representing the upper and lower faces of the prism.

You can see examples of triangular prisms in the two dormer windows on the roof of "Shakespeare's House." If you imagine a plane to extend horizontally, dividing the windows into two parts, the lower part in each

Shakespeare's House

case will be a triangular prism resembling the model given above. The bases are the vertical triangles close to the roof; they are still called "bases," although the prisms here do not rest upon them. The roof of the building forms the rectangular faces, and the front of the building and the imaginary cutting plane the square faces.

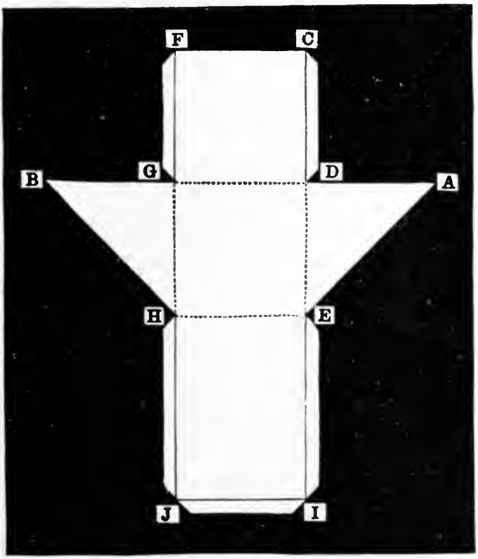
We will now make a model of the triangular prism.

2. The diagram will need paper 18 × 15 cm. (or $7\frac{1}{4} \times 6$ in.). AB is to be 1 dcm. 5 cm. (or 6 in.) long, and is divided into three equal parts at D and G, 5 cm. (or 2 in.) long.

CE and FH are each 1 dcm. (or 4 in.) long; they cross AB perpendicularly at D and G, where they are divided into equal parts.

When so much has been finished, *AE* and *BH* should be drawn, and then *CE* and *FH* should be prolonged so that *EI* and *HJ* may be equal to *AE* and *BH*.

Lastly, *CF* and *IJ* should be drawn.



3\. **1.** How many faces has this figure?

2. How many edges?

3. How many corners?

4. Are there any parallel faces? If so, how many pairs?

5. Are there any parallel edges? If so, how many groups?

6. What is the greatest number of parallel edges in any one group?

7. Are there any edges perpendicular to other edges? If so, what is the greatest number of edges which meet any one edge perpendicularly?

8. How many faces are there each bounded by four edges? Are those faces all equal to each other? What test would you apply?

9. How many edges bound each of the other faces? Are those faces equal? Test them.

10. Can you hold the figure so that six edges may be horizontal?

11. So that five edges may be horizontal?

12. So that three edges may be horizontal?

13. So that two edges may be horizontal?

14. So that two edges may be vertical?

15. So that three edges may be vertical?

4. **Variety in Prisms.** This figure is called a *prism*, which, as was said before, means "something sawed off;" that is, a prism is a part of another figure.

16. Can you see how your prism is a part of a cube? Can you place two of the prisms together so as to make a cube?

There is no limit to the varieties of prisms; but all prisms agree in having two faces parallel and equal to each other (these faces being bounded by any number of edges), and all other faces parallelograms.

17. Of what kind or kinds are the parallelograms in your prism?
18. The parallelograms may or may not be parallel to each other. How are they in your prism?
19. The parallelograms may or may not be equal to each other. How are they in your prism?

The parallelograms are called the *lateral* (lat′-er-al) or side faces of the prism; "lateral" means "on the side."

The two faces which must be parallel and equal are called the *bases* of the prism; and prisms take various names according to the shape of their bases, — rectangular, square, triangular, etc.

A *right* prism is a prism whose side faces are all squares or rectangles; here, "right" means "straight."

20. Of which kind is your prism?
21. Are parallelopipeds a variety of prisms?
22. If so, how many pairs of the faces may be called bases?
23. How does this differ from the case of other prisms?
24. As prisms are named according to the shape of their bases, what kind of a prism is a cube?
25. Is the cube a right prism?
26. What kind of a prism is a rectangular parallelopiped?
27. Is it a right prism?

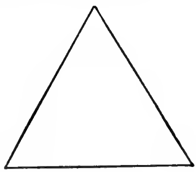
5. **Triangles.** Let us now examine the bases of the prism you have just made. How many edges bound each?

A face which is bounded by three edges is called a *triangle*.

There are several kinds of triangles; but all can be formed by cutting quadrilaterals into two parts from corner to corner.

An *equilateral* (e-qui-lat′-er-al) triangle is bounded by three equal edges.

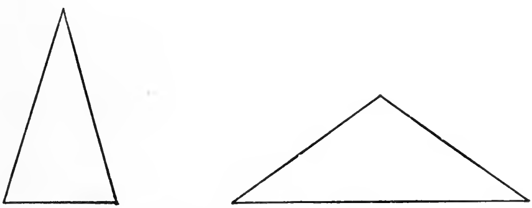
"Equilateral" means "having equal sides."



Equilateral

An *isosceles* (i-sos′-ce-les) triangle has two of its edges equal.

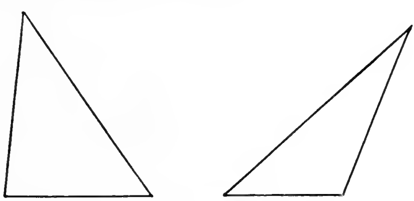
"Isosceles" means "having equal legs," the side which is not equal to the other two being called the "base."



Isosceles

A *scalene* (sca-lene′) triangle is bounded by three unequal edges.

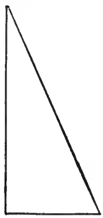
"Scalene" means "having crooked legs."



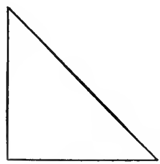
Scalene

An *oblique* (o-blique′) triangle has none of its edges perpendicular to one another. It may be equilateral, isosceles, or scalene. The preceding are examples also of oblique triangles.

A *right* triangle has two of its edges perpendicular to each other.



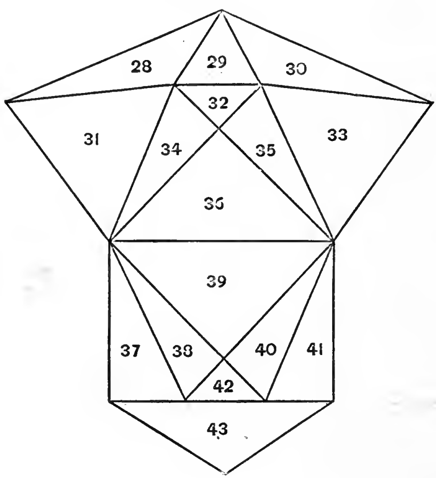
Right Scalene



Right Isosceles

A right triangle may be either scalene or isosceles. The edge which lies opposite the right angle is called the *hypotenuse* (hy-pot′-e-nuse).

In the following collection of triangles, give the name of each, first judging the forms by the eye, and then testing them by measuring the sides.

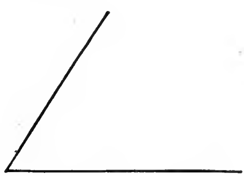


CHAPTER IV

ANGLES

1. NOTICE the hands of the clock in the illustration. The hour hand is horizontal, and the minute hand is vertical; the two are therefore at right angles with each other.

As the hands of a clock move, they are at right angles only twice during each hour; but at every instant they are making an angle of some kind with each other.



An Angle

An *angle* is a figure formed by two straight lines diverging from a point.

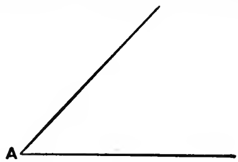
The word angle is derived from the Latin word *angulus*, meaning "a corner."

Belfry of Christ Church, Boston

The two lines are called the sides or legs of the angle.

The place where the lines meet is called the *vertex* (ver′-tex) of the angle.

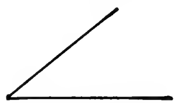
Vertex is a Latin word meaning "a top or turning point."



Vertex

The vertex is a *point*. A point has position only, — no length, breadth, or thickness.

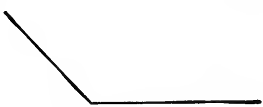
The size of an angle depends only upon the amount of inclination of one side to the other; it is not changed by lengthening or shortening the sides. The hands of a watch make countless different sizes of angles with each other in the course of an hour, but do not change their own length meanwhile; at three o'clock and nine o'clock the hands of a great clock and those of a small watch are alike perpendicular to each other, — that is, are at right angles.



An Acute Angle

An *acute* angle is less than a right angle.

Acute means "sharp."

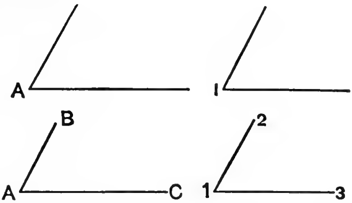


An Obtuse Angle

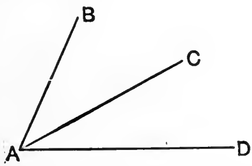
An *obtuse* angle is greater than a right angle.

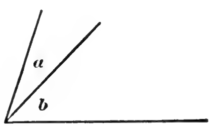
Obtuse means "blunt."

An angle may be designated by a letter or number placed near the vertex, or by three letters or numbers placed, one at the vertex, and one on each side of the angle.



If three letters are used, the one which indicates the vertex is placed between the other two in referring to the angle, as *BAC*. If no other angle has the same vertex, an angle is clearly designated by one letter; but if several angles have the same vertex, three letters are generally used so as to avoid confusion, or one letter can be placed between the sides of each angle.





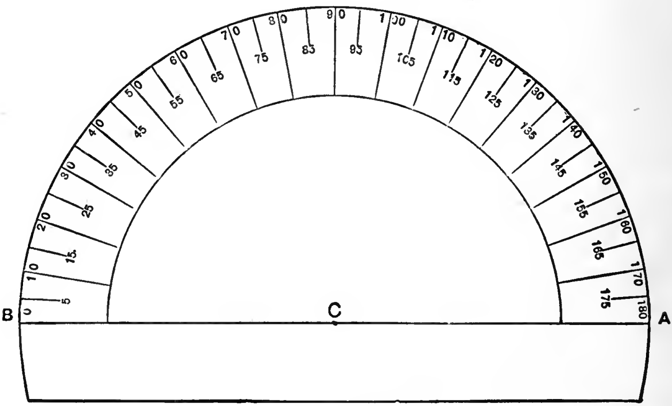
2. Table of the divisions of the right angle.

The right angle is divided into degrees (°), minutes (′), and seconds (″), according to the following table: —

60 seconds (″) = 1 minute (′)
60 minutes (′) = 1 degree (°)
90 degrees (°) = 1 right angle.

1. How do you read the angle 18° 27′ 43″?
2. 85° 14′ 30″?
3. 60° 20′ 48″?
4. Write with the signs; ten degrees, forty minutes, twenty seconds.
5. Thirty-eight degrees, seventeen minutes, six seconds.
6. How many degrees are there in two-thirds of a right angle?
7. In three-fourths of a right angle?
8. How many minutes are there in 37° 30′?
9. How many minutes are there in three-eighths of a right angle?
10. How many degrees are there in three-fifths of a right angle?

11. How many degrees are there in five-sixths of a right angle?
12. What part of a right angle is 18°?
13. What part is 60°?
14. What part is 72°?
15. What part is 80°?
16. What part is 22° 30′?
17. How many right angles are there in 120°?
18. In 108°?
19. In 135°?
20. In 126°?



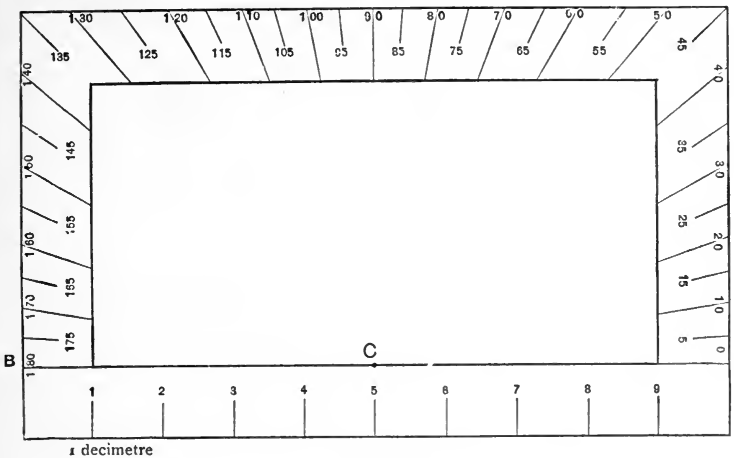
A Protractor

3. **Protractors.** A protractor is an instrument which is used to find the size of an angle, or to construct an angle of any required size. Protractors are made of thin metal, ivory, cardboard, etc., and in various shapes of which the commonest are shown in the annexed pictures. They may be graduated to any degree of minuteness; but intervals of five degrees are fine enough for our purpose.

If you have no protractor, you should make one out of card or stiff paper, copying one of the forms given here.

The lower straight edge can be graduated so as to serve for a measuring ruler.

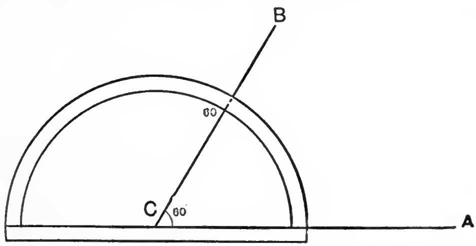
At the middle point of one edge, BA, of the protractor is a notch or dot, marked C in the diagram; this point is the vertex of any angle to



A Protractor

which the protractor is applied, and CA is placed directly on one side of the angle. The other side of the angle is indicated by the little lines at the edge of the protractor, having numbers which show the size of the angle in degrees. This second side is seldom drawn completely to the point C, because for convenience in use most protractors have an open space in the interior; but you will notice that if the lines around the rim were prolonged, they would all meet at the point C.

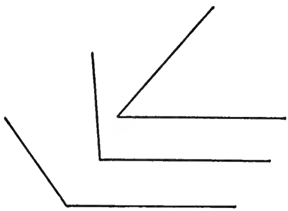
In the first picture of the protractor the angles are numbered from left to right; but in the second picture they are numbered from right to left according to the way in which angles are supposed to increase in size.



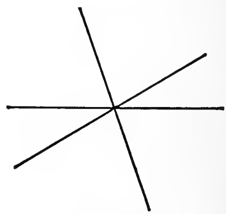
4. **To measure an angle, as ACB, with the aid of a protractor,** place the protractor as in the annexed figure, with the notch upon the vertex C, and the edge upon one side CA, so that the

point on the rim which indicates zero degrees may be on *CA*. Then observe the number of degrees marked on the rim of the protractor where it is crossed by the other side, *CB*, of the angle. This will be the number of degrees in the angle, if the protractor is graduated from right to left; but if from left to right, the number on the rim must be subtracted from 180° to show the size of the angle.

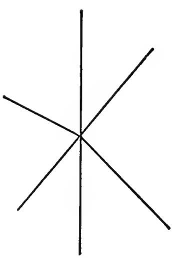
Estimate by the eye the size of the following angles, and then test your answers with a protractor.



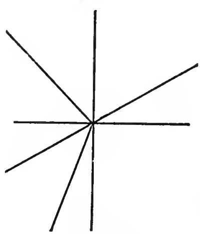
21–23



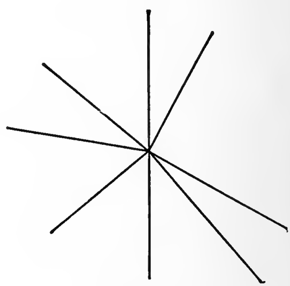
24–29



30–35

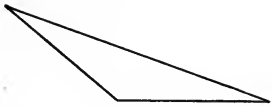


36–43

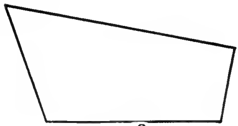


44–51

5. **To construct an angle of required size with the aid of a protractor.** Suppose you wish to construct an angle of 140°. Draw a straight line *CA* of any convenient length. Place the protractor with its notch at *C* and the edge along *CA*. Then find on the rim of the protractor the mark which indicates the

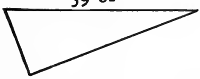


52–54



55–58

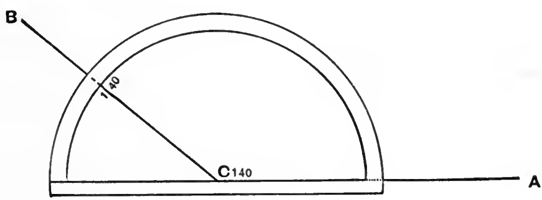
59–61



62–65

66–70

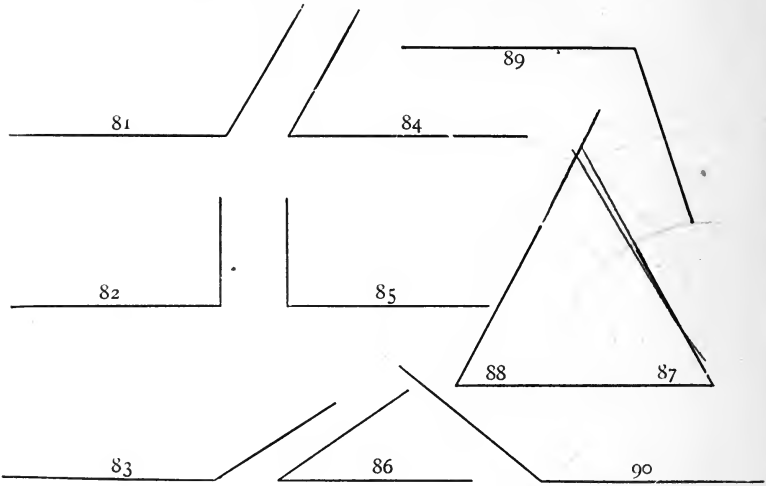
angle 140°. Dot the paper at that point, remove the protractor, and draw the line *CB* through the dot. *ACB* will be the required angle.



Construct the following angles with the aid of a protractor:—

71. 60°.	**75.** 155°.	**78.** 85°.
72. 160°.	**76.** 170°.	**79.** 105°.
73. 45°.	**77.** 25°.	**80.** 5°.
74. 80°.		

Construct angles equal to the following with the aid of a protractor:—



Construct the following angles, ruling the lines but otherwise aided by the eye alone: then test your angles with a protractor: —

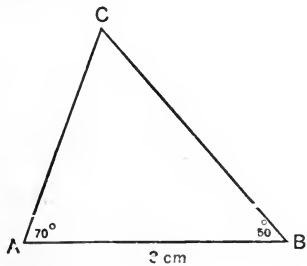
91. 30°.
92. 120°.
93. 45°.
94. 135°.
95. 90°.
96. 50°.
97. 130°.
98. 20°.
99. 100°.
100. 85°.
101. 40° and 140°, to have their vertices at the same point and one side in common.
102. 130° and 50°.
103. 90° and 90°.
104. 60°, 90°. 120°, 90°, to have their vertices at the same point.
105. 45°, 135°, 80°, 100°.

CHAPTER V

CONSTRUCTIONS OF SOME PLANE FIGURES

1. To construct a triangle when you know the length of one side and the size of the angles at the ends of that side.

Suppose the side to be 3 cm. long and the angles at the ends of the side to be 70° and 50°.



Draw the line AB 3 cm. long.

At A draw a line making with AB an angle of 70°, and at B a line making with AB an angle of 50°, so that the two lines may meet at C.

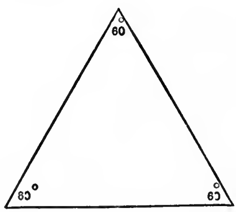
Then ACB will be the required triangle.

Construct triangles having the following sides and angles:—

1.	Side	5 cm ;	angles	60°	and	60°.
2.	"	5 cm. ;	"	90°	"	45°.
3.	"	3 cm. ;	"	70°	"	70°.
4.	"	4 cm. ;	"	100°	"	30°.
5.	"	3 cm. ;	"	100°	"	50°.
6.	"	2 inches ;	angles	60°	and	60°.
7.	"	3 "	"	30°	"	45°.
8.	"	2 "	"	45°	"	45°.
9.	"	2 "	"	90°	"	45°.
10.	"	2 "	"	70°	"	50°

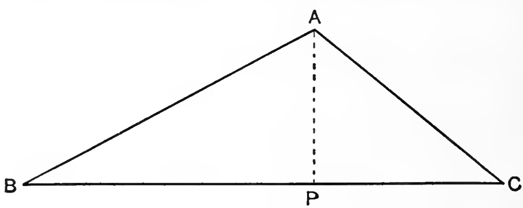
2. The triangle in which the two angles are each 60° should be carefully observed. If you measure the third angle, you will find that it is 60°; and if you measure the two sides of this angle, you will find that they are each of the same length as the first side. This triangle, therefore, is both equiangular (that is, "having all three angles equal") and equilateral (that is, "having all three sides equal").

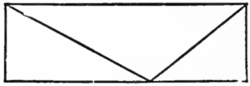
If you wish to construct an equilateral triangle with a side, say 5 cm. long, you can draw a line 5 cm. long, and at each end make an angle of 60°, prolonging the lines until they meet. The resulting triangle will be both equilateral and equiangular.



An Equilateral and Equiangular Triangle

3. **The sum of the three angles of any triangle** is 180°, which is equal to two right angles. You can test this by an experiment.

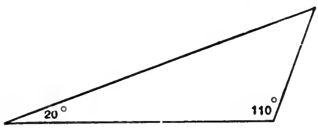




Draw a triangle ABC of any shape and size, and draw a perpendicular AP upon one of the longer sides, BC, thus forming two right angles, APB

and APC. Cut out the triangle from the paper, and fold over the three vertices upon P. You will see that the three angles of the triangle exactly cover the two right angles beneath.

If, therefore, you know the size of two angles of a triangle, you can find the third angle by subtracting their sum from 180°.



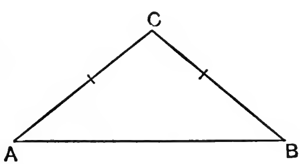
Thus if two angles of a triangle are 20° and 110°, their sum is 130°, which leaves 50° for the third angle.

Find the number of degrees in the third angle of the following triangles:

11. A = 20°, B = 40°.	**16.** A + B = 100°.
12. A = 80°, B = 60°.	**17.** A + B = 140°.
13. A = 30°, B = 130°.	**18.** A + B = 10°.
14. A = 45°, B = 90°.	**19.** A + B = 95°.
15. A = 70°, B = 70°.	**20.** A + B = 175°.

4. Besides the equilateral triangle, two others need special notice, — the isosceles and the right triangle.

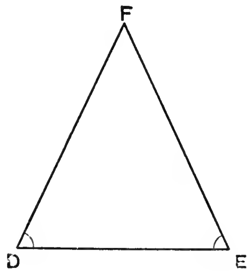
In an isosceles triangle there are always two equal angles, which lie opposite the equal sides, at the ends of the base. The third angle is called the *vertex* angle.



Thus in the triangle ABC, in which CA and CB are equal, the angles A and B are equal to each other; and C is the vertex angle.

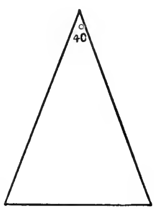
Also, if you know that two angles of a triangle are equal, you can infer that two sides are equal, and the triangle is isosceles.

Thus in the triangle *DEF*, if the angles *D* and *E* are known to be equal, the sides *FD* and *FE* are also equal.

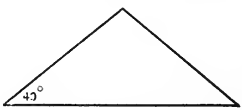


If, therefore, you are told the size of one angle of an isosceles triangle, you need only to know whether this angle is the vertex angle or one of the equal ones, in order to find the sizes of the other two.

Suppose, for example, that the vertex angle of an isosceles triangle is 40°. Subtracting 40° from 180°, you would have 140° left for the sum of the other two angles; and as they are equal, each must be 70°.



If one of the equal angles of an isosceles triangle is 40°, another angle is also 40°; these two together contain 80°, which leaves 100° for the vertex angle.



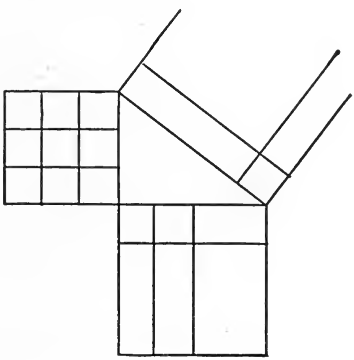
Find the size of each angle of the following triangles, knowing that the triangles are isosceles and that the given angle is the vertex angle: —

21. 20°.
22. 40°.
23. 150°.
24. 80°.
25. 70°.
26. 45°.
27. 90°.
28. 140°.
29. 85°.
30. 15°.

Find the size of each angle of the following triangles, knowing that the triangles are isosceles and that the given angle is one of the equal angles: —

31. 30°.
32. 70°.
33. 25°.
34. 80°.
35. 15°.
36. 50°.
37. 35°.
38. 75°.
39. 10°.
40. 85°.

5. **A right triangle** has one very important property which we will now investigate.

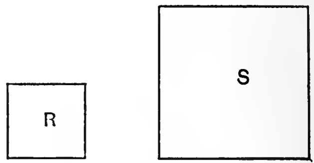


41. Draw a right angle with sides 3 cm. and 4 cm. (or $\frac{3}{4}$ in. and 1 in.) long, and complete a right triangle by drawing the hypotenuse.
42. Upon each side of the triangle draw a square.
43. Divide each square into smaller ones having sides 1 cm. (or $\frac{1}{4}$ in.) long, and count the squares.
44. How does the number of squares formed on the hypotenuse compare with the sum of the squares on the other two sides?
45. Do the same as you have done with the previous triangle, making the edges of the right angle 5 cm. and 12 cm. (or $1\frac{1}{4}$ in. and 3 in.) long.

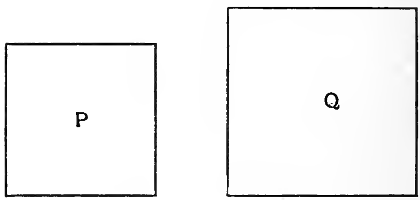
The connection between the area of the square on the hypotenuse and the sum of the areas of the squares on the other two sides is the same in any right triangle as in the case of the two which you have just drawn, and is expressed as follows: The square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides.

To construct a square, therefore, whose area shall be equal to the sum of the areas of two other squares, you have only to draw a right triangle with the sides of the right angle equal to sides of the given squares, and then draw a square on the hypotenuse; this will be the required square.

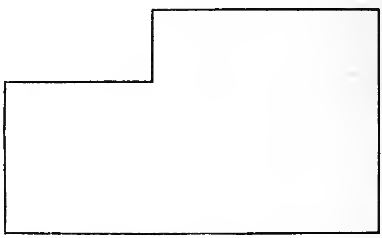
46. Draw a square whose area shall be equal to the sum of the areas of the squares *R* and *S*.



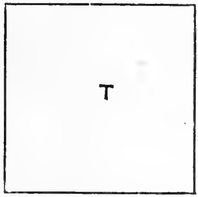
47. Draw a square whose area shall be equal to the sum of the areas of the squares *P* and *Q*.



48. The annexed figure is composed of two squares. Copy the figure on paper, but draw each line twice as long as in the diagram. Then draw a line between two of the corners, which shall be the side of a square having the same area as your diagram.

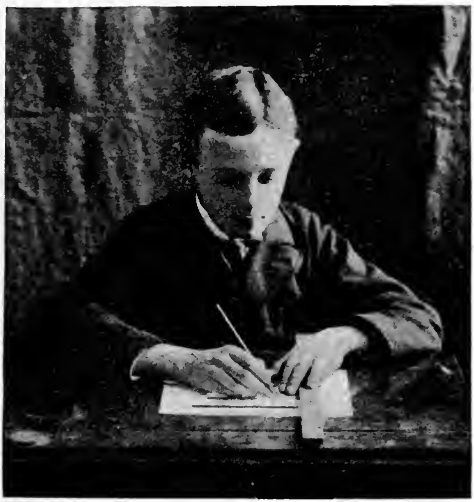


49. Draw a square whose area shall be twice that of the square *T*.



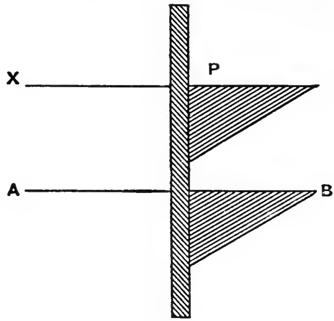
50. A man owns two pieces of land, one 6 rods square and the other 8 rods square, which he is to exchange for a single piece of land also square-shaped and of as great an area as the other two together. What is the length of a fence which will enclose his new land?

6. To draw a straight line through a given point parallel to a given straight line.



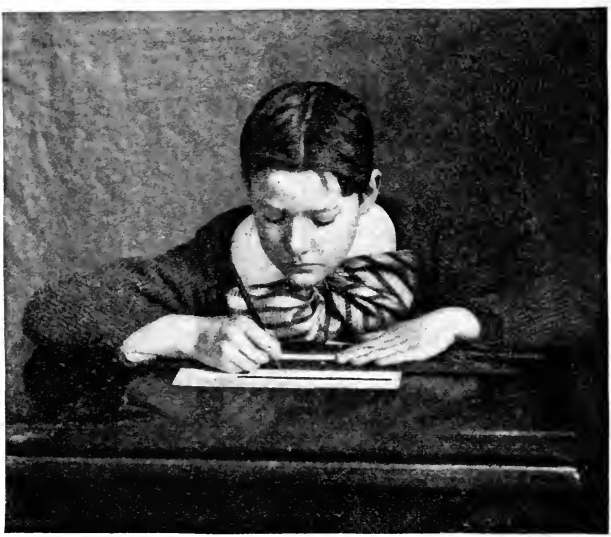
Using a Ruler and Square.

(*a*) With the aid of a ruler and square. Suppose you wish to draw through the point *P* a line parallel to *AB*.



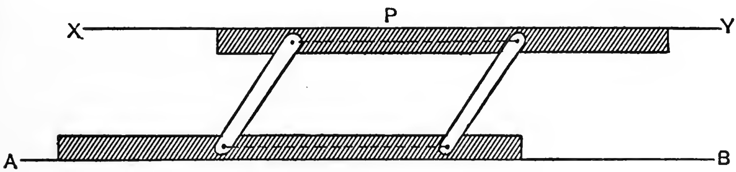
Place the ruler and square so that the edge of the ruler may be close to P, and the square may have one edge close to AB and the other against the ruler. Then slide the square along the ruler until it reaches P. Along the edge draw PX, which will be the required line parallel to AB.

(*b*) With the aid of a " parallel ruler."



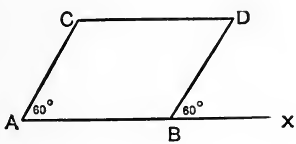
Using a Parallel Ruler

This instrument consists of two rulers fastened together with two strips of metal swinging on pivots at each end. The distances between the pivots, measured along the metallic strips, are equal; and the distances between the pivots, measured along the rulers, are also equal. Thus the pivots are the vertices of a parallelogram whether the ruler is open or shut. All four edges of the ruler are made parallel to each other so that parallel lines may be ruled along any of them.



To draw a straight line through P parallel to AB, place the ruler with one edge close to AB, and hold that half of the instrument firmly in its place. Swing the other half on the pivots until its edge reaches P; then draw XY along that edge through P; this will be the required line parallel to AB.

7. To construct a parallelogram when you know the lengths of two sides which meet and the size of the angle between them.



Suppose the sides to be 4 cm. and 3 cm. long, and the angle to be 60°.

Draw the line AB, 4 cm. long.

At A draw AC, 3 cm. long, making with AB an angle of 60°.

Prolong AB through B some convenient length to X.

Draw BD of the same length as AC and making the angle XBD of the same size as the angle A.

Draw the straight line CD.

Then $ABCD$ will be the required parallelogram.

BD could also be drawn with the aid of a square or a parallel ruler.

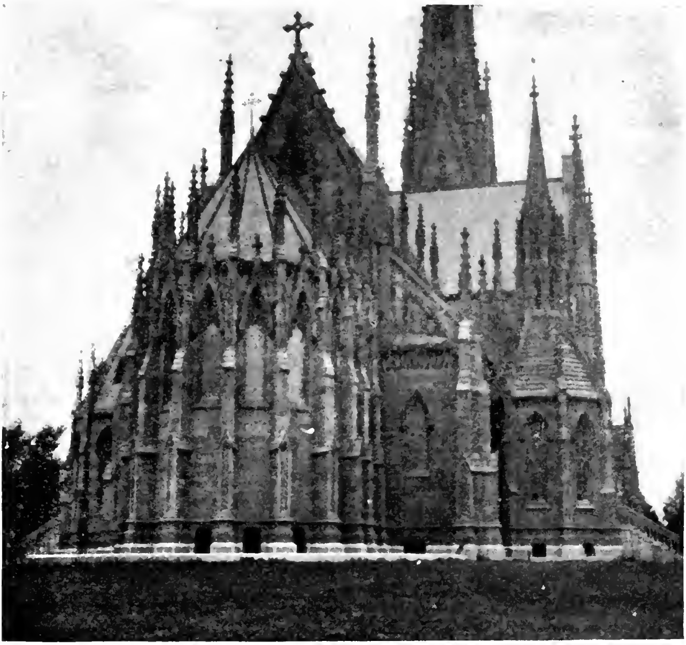
Construct parallelograms having the following sides and angles; and then tell the kind of parallelogram each is:—

51.	Sides,	5 cm.	and	2 cm.;		angle	45°.	
52.	"	5 cm.	"	5	"	"	60°.	
53.	"	4 cm.	"	3	"	"	90°.	
54.	"	3 cm.	"	3	"	"	90°.	
55.	"	2 inches	and	3	inches;	angle		50°.
56.	"	2	"	" 2	"	"		120°.
57.	"	2	"	" 2	"	"		90°.
58.	"	2	"	" 1	inch	"		90°.

The sum of the three angles of any triangle is two right angles, or 180°.

The square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides.

CHAPTER VI



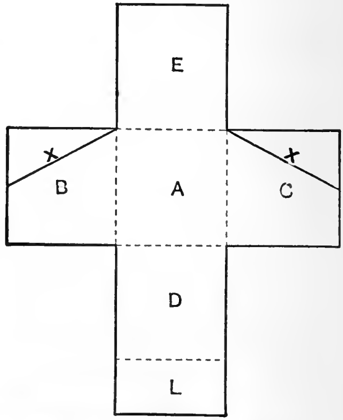
Cathedral of the Incarnation, Garden City, N. Y.

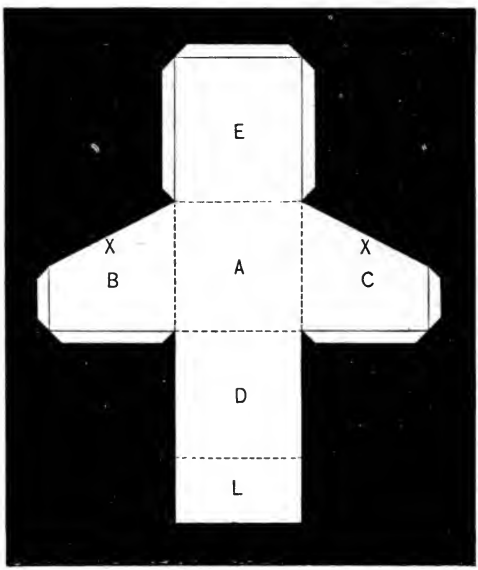
A TRUNCATED PRISM

OBSERVE the buttresses on the church in the picture. They are not prisms such as we have been studying, for their upper surfaces are inclined to their bases, but they are what are called *truncated* prisms, truncated meaning "lopped off." A

truncated prism is one from which a part has been cut off by a plane inclined to the base.

We will now make a model of a truncated square prism or cube.





The diagram will need paper 19 cm. × 17 cm. (or $7\frac{1}{2}$ in. × $6\frac{1}{2}$ in.).
The construction can be seen from the special figure.
A, *B*, *C*, and *D* are squares with edges 5 cm. (or 2 in.) long.
L is a rectangle with the shorter edges 2 cm. 5 mm. (or 1 in.) long.
From two corners of *A* lines *X* are drawn to the middle points of the outer edges of two adjoining squares.
E is a rectangle with the longer edges equal to *X*.

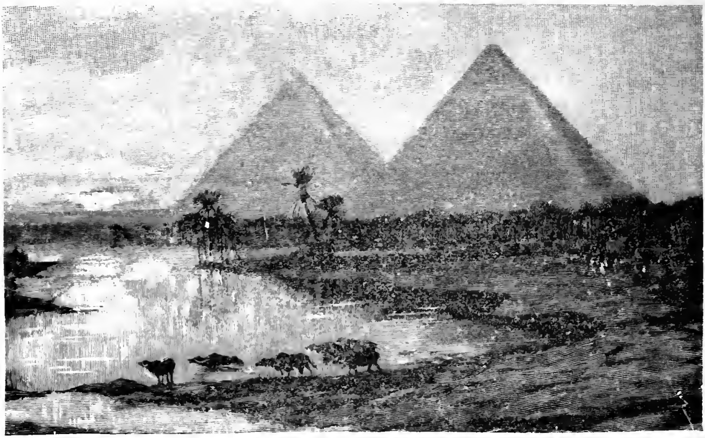
The base of the truncated prism is the base of the prism of which it was a part.

The face formed by the cutting plane is called the *inclined section.*

The other faces are called lateral or side faces.

1. What are the shapes of the lateral faces of this figure?
2. If you were to place the figure upon one of its lateral faces as a base, what should you then call the figure?
3. How does it happen that this figure has different names according to its position?
4. Supposing the original figure to have been a cube, can you see what the shape of the part cut off must have been?
5. Can you place two of these truncated prisms together so as to form a rectangular parallelopiped?
6. What would be the volume of that parallelopiped?
7. What, then, is the volume of your truncated prism?

CHAPTER VII



The Pyramids of Egypt

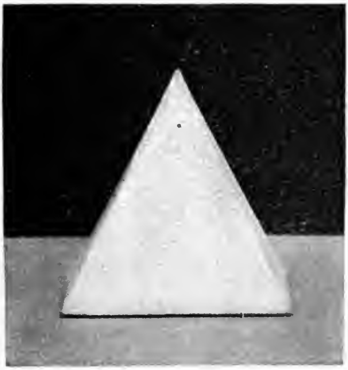
A PYRAMID

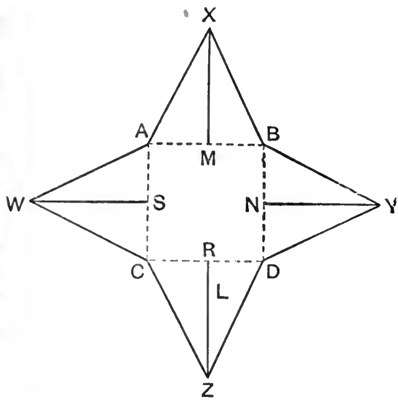
1. IN the above picture you see a very ancient geometric form, supposed to have been invented by the Egyptians. This is the *pyramid* (pyr′-a-mid), a word whose original meaning is uncertain, though some writers trace a connection between it and a word meaning "fire."

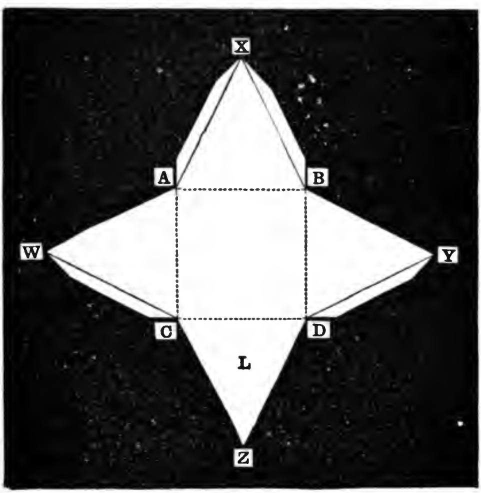
A pyramid has all its faces, except one, triangles which meet at a point called the *apex* (a′-pex), a word meaning "a top or summit." The other face, which may have any number of edges, is called the *base;* and a pyramid takes a

name — square, triangular, etc. — according to the shape of its base.

We will now make a model of a pyramid having a square base.







2. The diagram will need paper 16 cm. 5 mm. × 16 cm. or $6\frac{1}{2}$ in. × $6\frac{1}{2}$ in.
The construction can be seen from the special figure.

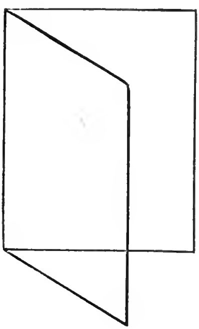
First draw a square with edges 5 cm. (or 2 in.) long, and find their middle points, *M*, *N*, *R*, and *S*.

Then draw outward from the edges the perpendicular lines *MX*, *NY*. *RZ*, and *SW*, each 5 cm. 6 mm. (or $2\frac{1}{4}$ in.) long.

Then draw *XA*, *XB*, *YB*, etc., to the corners of the square.

3. **1.** How many faces has this pyramid?
2. How many edges?
3. How many corners?
4. How many angles have the faces altogether?
5. How many of the edges are perpendicular to other edges?
6. What is the greatest number of edges perpendicular to any one edge?

4. **Diedral Angles** You have seen how the edges of faces may make angles with other edges. You will now notice that the faces themselves may make angles with other faces, and in fact always do if they are extended far enough, unless they are parallel. But notice that instead of meeting at a point as two edges do, two faces meet in a straight line.



An angle formed by two faces is called a *diedral* (di-e′-dral) angle.

Diedral means "having two sides."

Diedral angles, like line angles, may be acute, right, or obtuse.

7. How many diedral angles does the base of the square pyramid form with the other faces?
8. Do these angles seem to you to be acute, right, or obtuse?
9. How many diedral angles are made by the triangular faces among themselves?
10. Of what kind do these angles seem to you to be?

Diedral angles may be measured with the aid of a stiff rectangular card, five or six inches long and two inches wide, folded so that the shorter edges may be exactly together.

The card is placed with its folded edge against the edge of the angle to be measured, the halves of the card lying

Measuring a Diedral Angle

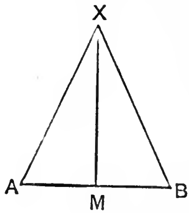
closely upon the faces of the angle. The protractor is then placed with the notch at one end of the folded edge of the card, and the angle between the two diverging edges of the card is measured: this angle is equal to the diedral angle of the faces.

You should now practise measuring the diedral angles between the faces of the other figures you have made.

5. **Area of a Triangle.** The surface of your pyramid is composed of a square base and four triangles. You know already how to find the area of the base; and we will now examine the other faces.

These faces are triangles. The area of a triangle is equal to the length of any one of its sides multiplied by one-half the perpendicular distance of that side from the opposite vertex.

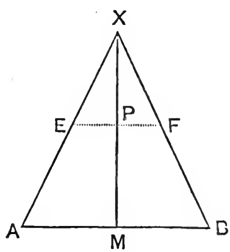
We will first calculate the area of one of the triangles from the diagram which you used in constructing the pyramid, and then test the answer by measurement.

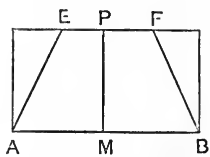


In the triangle AXB, what length is represented by AB?
What length is represented by XM?
How much is one-half the length of XM?
Multiplying one-half of XM by AB gives what for the area of the triangle?

Now let us test this by measurement.

Construct on paper a triangle of the exact size of one of the faces: — Draw AB, 5 cm. (or 2 in.) long, and find the middle point M.





At M draw the perpendicular MX, 5 cm. 6 mm. (or $2\frac{1}{4}$ in.) long, and then draw XA and XB.

Cut the triangle carefully out from the paper.

Fold over the upper part so that X may come exactly upon M, and make the crease represented by EPF. Cut off the part EXF along the crease, and cut this again into two parts along the line XP. Then match these two parts upon the rest of the triangle as shown in the figure.

You have now turned the triangle into a rectangle, which you can paste together with a strip of paper on the back.

11. What is the length of this rectangle?
12. What is the width?
13. What is the area?
14. Does this result agree with the answer you obtained by calculation from the diagram? If not, see if you can find where you have made a mistake. The area is 14 sq. centimetres, or, if you have used English measurements, $2\frac{1}{4}$ sq. inches.
15. What is the area of the four triangles together?
16. What is the area of the entire surface of the pyramid?

6. **The volume of a pyramid** is equal to one-third of its height multiplied by the area of its base.

We will test this by trying an experiment with the pyramid and cube.

Testing the Height of a Pyramid

First, place the base of the pyramid against the base of the cube, and see that they have the same area. Then set the two figures on a horizontal plane at a little distance apart, and rest a ruler across the top of the cube and apex of the pyramid: see if the ruler is horizontal; you will find it to be almost exactly so if you have made the two figures correctly. So the heights of the cube and pyramid are equal, as well as the bases. Now the volume of the cube is equal to the area of its base

multiplied by the whole of its altitude; so if the volume of the pyramid is one-third of this, the pyramid ought to hold only one-third as much as the cube.

Make a new pyramid, therefore, so as to save the other, but before pasting the last edge cut off the square base. Then take the cube which you keep for measuring, and using sand or water as you did before, see if the pyramid has to be filled three times in order to fill the cube once.

17. How many cubic centimetres, therefore, are there in the volume of your pyramid?

18. How many fillings of your pyramid would make one of the parallelopiped described on page 17?

19. If the edge of the base of your pyramid were twice as long as it is, what would be the volume?

20. If the base were of the same size as it is, but the height were twice as great, what would be the volume?

21. What would be the volume of a pyramid with a height 6 inches and a base containing 9 square inches?

22. If a pyramid and a cube have equal bases containing 16 square inches, what must be the height of the pyramid so that the volumes of the two figures may be equal?

23. How many pyramids, each with a height of 3 cm. and a base area of 16 sq. cm., could be filled from the contents of a rectangular parallelopiped $4 \times 6 \times 8$ cm.?

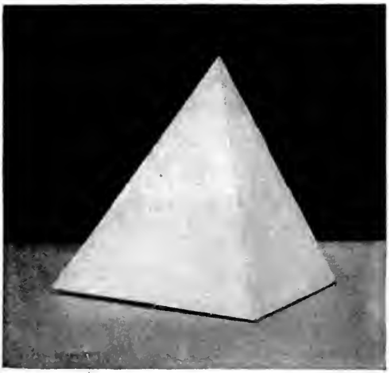
The area of a triangle is one-half the product of its base and altitude.

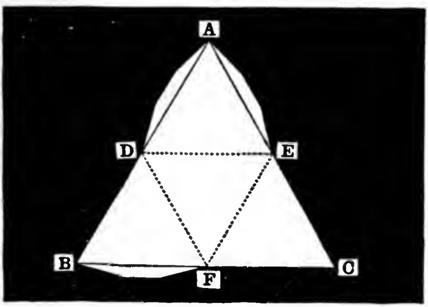
$$\textit{Area triangle} = \frac{\textit{base} \times \textit{altitude}}{2} = \frac{\textit{base}}{2} \times \textit{altitude} = \textit{base} \times \frac{\textit{altitude}}{2}$$

The volume of a pyramid is one-third the product of the area of its base and altitude.

$$\textit{Volume pyramid} = \frac{\textit{base} \times \textit{altitude}}{3} = \frac{\textit{base}}{3} \times \textit{altitude} = \textit{base} \times \frac{\textit{altitude}}{3}$$

CHAPTER VIII





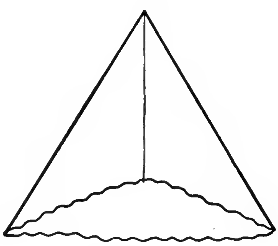
A TRIANGULAR PYRAMID

1. THE diagram will need paper 10 cm. × 11 cm., or 4 in. × 4½ in. *ABC* is an equilateral triangle, each edge being 1 dcm. long. The middle points of the edges, *D*, *E*, and *F*, are joined, and the triangle is thus divided into four smaller equilateral triangles having edges 5 cm. long.

In English measurements, the edges of *ABC* may be 4 inches long, thus making the edges of the smaller triangles 2 inches long.

2. **1.** How many faces has this figure? What is their shape?
2. How many edges? What is their length?
3. How many corners?
4. How many line angles? What is their size?
5. How many diedral angles? What is their size?
6. This figure is called a pyramid: why?
7. It is also called a triangular pyramid: why?
8. Has it more than one face which can be called its base?
9. Has a quadrangular pyramid more than one such face?
10. Can you explain the difference between these two cases?

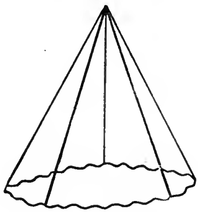
3. **Solid Angles.** We have seen that when two edges meet, or would meet if prolonged, they form a line angle; and when two faces meet, or would meet if extended, they form a diedral angle. Now when three or more faces meet at a point and enclose all the space around the point, they make what is called a solid angle.



A Solid Angle

If you observe figures carefully you will see that it takes at least three faces to form a solid angle; for two faces would leave an open space. But there may be as many faces as you please more than three; though if you try to make a solid angle by joining pieces of paper you will find that the sum of the angles formed by the edges must not be so great as 360° or 4 right angles. If the sum were equal to 360°, the pieces of paper would lie flat and form a plane.

Notice that a solid angle has an open space in front of the point. If this space were closed by a plane cutting the other faces, the resulting figure would be a pyramid.



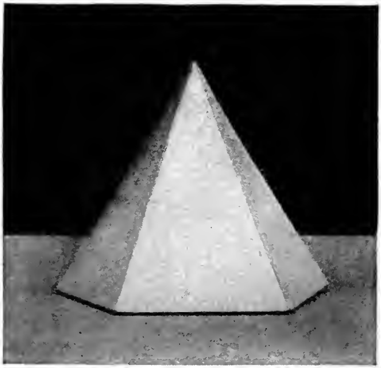
A Solid Angle

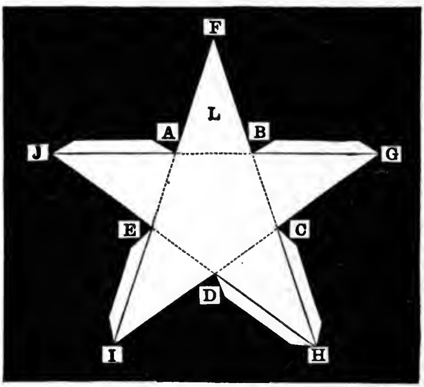
If a solid angle is formed by three faces, it is called a *triedral* (tri-e′-dral) angle, which means "having three faces."

If it is formed by four or more faces, it is called a *polyedral* (pol-y-e′-dral) angle, which means "having many faces."

11. What is the difference between a triedral angle and a triangular pyramid?
12. How many solid angles has a triangular pyramid?
13. How many has a cube? What is the sum of the line angles which form each solid angle?
14. How many solid angles has a quadrangular pyramid?
15. Is there a solid angle at each corner of a figure which is entirely enclosed by planes?
16. In the triangular pyramid is the number of solid angles equal to the number of faces?
17. Is this true of the cube?
18. Of the triangular prism?
19. Of the quadrangular pyramid?

CHAPTER IX





A PENTAGONAL PYRAMID

1. The diagram will need paper 15 × 15 cm., or 6 × 6 inches.
Draw AB, 3 cm. long.
At A draw AE, 3 cm. long, and making the angle BAE 108°.

At B draw BC, 3 cm. long, and making the angle ABC 108°.

At E draw ED, 3 cm. long, and making the angle AED 108°.

Draw the line DC completing the interior part of the diagram.

Prolong the lines AB, BC, etc., in both directions until they form the five-pointed star $FGHIJ$.

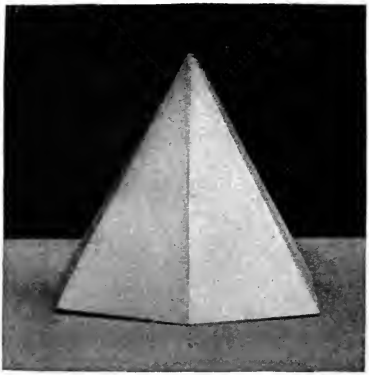
With English measurements, 1 inch will be a convenient length for AB.

2. This figure is called a *pentagonal* (pen-tag′-o-nal) pyramid. Its base is a pentagon (pen′-ta-gon), which means a face having five corners. It also has five edges; for every face has as many edges as it has corners.

3. Examine the completed model, make measurements, and write out your answers to the following: —

1. The number of faces.
2. The number of edges.
3. The number of corners.
4. The shapes of the faces and the number of each shape.
5. The lengths of the edges and the number of each length.
6. The number of face angles.
7. The sizes of the face angles and the number of each size.
8. The number of diedral angles.
9. The sizes of the diedral angles and the number of each size
10. The number of solid angles.
11. The number of faces which form each solid angle.

CHAPTER X



A HEXAGONAL PYRAMID

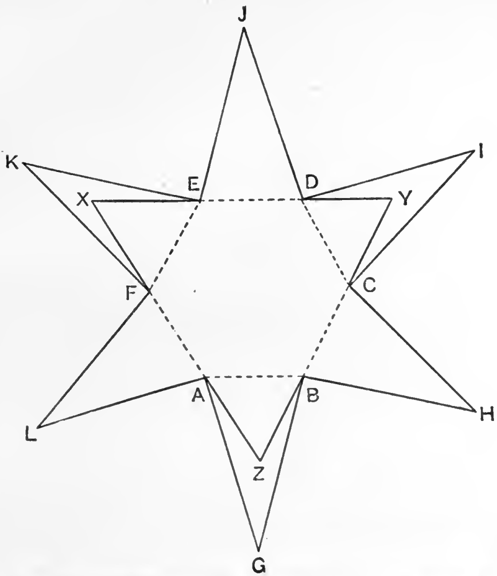
1. The diagram will require paper 20 × 20 cm. or 8 × 8 inches.

Construct the equilateral triangle *XYZ*, each side 9 cm. (or $4\frac{1}{2}$ in.) long.

Divide each side into equal parts, each 3 cm. (or $1\frac{1}{2}$ in.) long, the points of division being *A*, *B*, *C*, *D*, *E*, and *F*.

Draw *AB*, *CD*, and *EF*, thus completing the interior part of the diagram *ABCDEF*, which has six dotted sides all equal.

Upon each of the six sides *AB*, *BC*, *CD*, etc., construct an isosceles triangle with the angles at *A*, *B*, *C*, etc., each 75°, thus forming the six-pointed star *GHIJKL*.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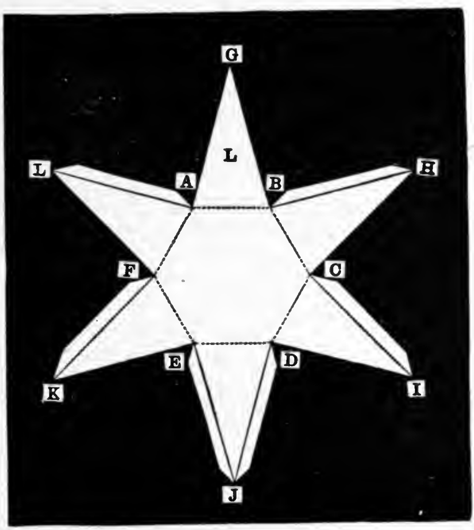
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CHAPTER XI

POLYGONS AND SYMMETRY

1. YOU have been told that pyramids take their names from the shape of their bases. Now the bases, like all faces, take their names as follows: —

First, from the number of edges or corners, the number of edges being the same as the number of corners.

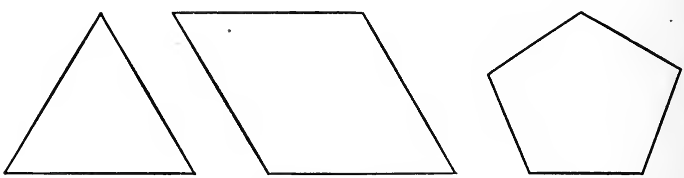
Secondly, from equality in the length of the edges.

Thirdly, from equality in the size of the angles.

Fourthly, from equality in both edges and angles.

Fifthly, from peculiarity in the arrangement of the edges or angles.

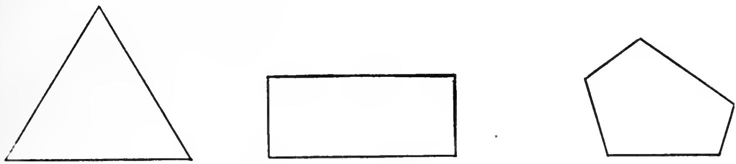
The general name for a face is *polygon* (pol′-y-gon), which means "having many corners;" but this name is usually applied only to faces which have more than four corners, that is, more than four edges.



Equilateral Polygons

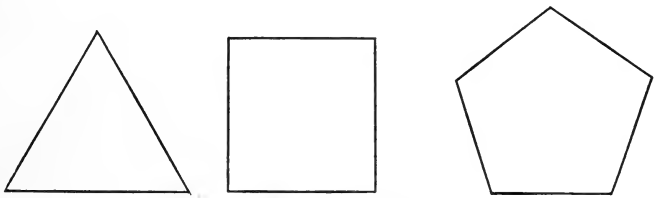
If the edges of a face are all equal to each other, it is called an *equilateral polygon*.

If the angles of a face are all equal to each other, it is called an *equiangular polygon.*



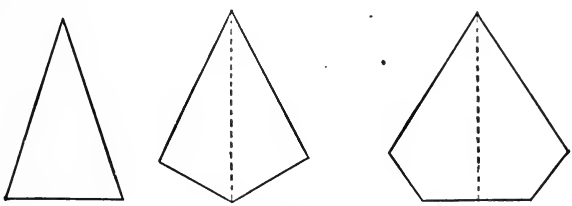
Equiangular Polygons

If a face is both equilateral and equiangular, it is called a *regular polygon.*



Regular Polygons

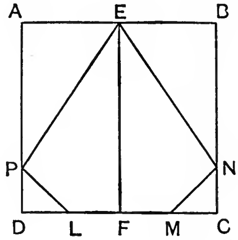
2. **A polygon is symmetrical with respect to a straight line** when this line divides it into two parts such that, if the figure



Symmetrical Polygons

be revolved on the line, the two parts will exchange places, each exactly covering the space formerly occupied by the other. The straight line is called the *axis of symmetry.*

You can test this by an experiment. First construct a symmetrical polygon as follows: Draw a square $ABCD$ with edges 4 cm. (or 2 in.) long.



Draw EF connecting the middle points of two opposite edges AB and DC. Divide each edge into four equal parts.

Draw PL and MN connecting the points of division nearest D and C.

Draw EP and EN.

A symmetrical polygon $LMNEP$ will thus be formed, of which EF is the axis. Cut out this polygon, using a ruler and knife so as to preserve the edges of the gap left in the paper. Then turn the polygon over, and replace it in the paper in the reversed position. You will see that the ends of the axis EF are in their former position; but N and P, and M and L, have exchanged places; thus all points in the polygon, except those in the axis, have exchanged places with other points.

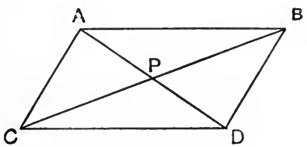
Draw the following figures, all of which are symmetrical with respect to a line, and then draw their axes: —

1. An isosceles triangle.
2. A straight line.
3. An angle with equal sides.
4. An equilateral triangle (three axes).
5. A square (four axes).
6. A straight line met at its middle point by two equal straight lines so as to form three angles each 60°.
7. A rectangle (two axes).
8. A parallelogram with equal angles.
9. A rhombus (two axes).
10. A trapezoid with two equal sides.

Two figures when considered together may be symmetrical with respect to a line.

For example, you can draw a polygon in ink, and before the ink is dry you can fold the paper so as to make an impression on some other part. Then the polygon and its impression will be symmetrical with respect to the crease in the paper which will represent the axis.

3. **A figure is said to be symmetrical with reference to a point** when, being turned half-way round on this point as a pivot, it exactly covers every part of the surface which it occupied in its former position. The pivot point is called the *centre of symmetry*. In this case a figure keeps in its own plane throughout its revolution; whereas, when it revolves on an axis, it leaves the plane at once and returns to it only when the revolution is completed.



You can test this by an experiment.

Draw a parallelogram *ABCD*, and connect its opposite corners with straight lines; the point *P* where these lines cross will be the point of symmetry. Cut out the figure with a knife, preserving the edges carefully. Return the figure to its place, and stick a pin through the pivot point. Then revolve the figure about *P* until the gap in the paper is filled again. You will see that every point except the pivot has moved, each exchanging places with another which was equally distant from the pivot; thus *A* changes places with *D*, and *B* with *C*.

Draw the following figures, all of which are symmetrical with respect to a point, and indicate the point in each case by the letter *P*:—

11. A straight line.

12. A square.

13. A straight line with two equal perpendicular lines, one from each end, extending in opposite directions.

14. A rhombus.

15. Two unequal straight lines cutting each other perpendicularly into equal parts.

16. Two equal parallel lines.
17. Two unequal straight lines cutting each other into equal parts, but not perpendicularly.
18. A rectangle.
19. A straight line from the ends of which extend two equal lines on opposite sides of the first line with which each makes an angle of 60°.
20. A straight line at the ends of which are two parallel equal lines which the first line divides into equal parts.

Two figures when considered together may be symmetrical with respect to a point.

Cut out a polygon of any shape. Make a tracing of this on paper; then turn the polygon half-way around so that one edge may be in a straight line with its former position, and make another tracing. The two tracings combined will be symmetrical with respect to a point half-way between the two nearest vertices.

In the preceding examples we have what is called twofold symmetry with reference to a point. An equilateral triangle is an example of threefold symmetry; in this case the figure when revolved only one-third of the way around the point occupies the same surface it covered at first, and after three revolutions it returns to its original position. In the same way the base of the pentagonal pyramid of Chapter IX. has fivefold symmetry. All regular polygons have a symmetry of as many fold as they have sides. Furthermore, a figure may have symmetry of several kinds; thus the base of the hexagonal pyramid of Chapter X. has two, three, and six fold symmetry.

Of how many fold is the symmetry of the figures in Chapter XXV. 13, of Part II., as follows? —

21.	Problem	1.	**25.**	Problem	16.
22.	"	5.	**26.**	"	20.
23.	"	11.	**27.**	"	24.
24.	"	14.	**28.**	"	25.

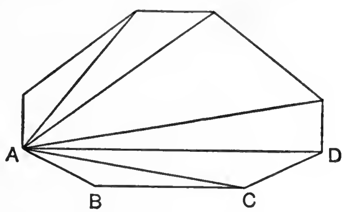
29. Are your hands symmetrical with respect to a point or to a line?
30. Of which kind is the symmetry of flowers such as the clematis and narcissus?
31. In which kind of symmetry do leaves usually appear on a branch?

4. **The perimeter** (pe-rim′-e-ter) of a polygon is the sum of the edges which bound it.

The word means "a measuring around."

Calculate the perimeters of the following figures: —

32. A rhombus whose edge is 5 cm.
33. A rectangle whose length is 5 cm. and width 3 cm.
34. A square, two of whose edges combined are 8 cm. long.
35. A parallelogram two of whose edges are 3 cm. and 7 cm.
36. A parallelogram in which the distance from one corner to the opposite corner, measured along the edges, is 12 cm.
37. An equilateral triangle, one of whose edges is 4 cm.
38. An equilateral triangle, two of whose edges combined are 10 cm. long.
39. An equilateral polygon bounded by eight edges, one of which is 2 cm.
40. An equilateral polygon bounded by twelve edges, five of which combined are 15 cm. long.



5. **A diagonal** (di-ag′-o-nal) of a polygon is a straight line drawn between any two corners which are not already connected by an edge.

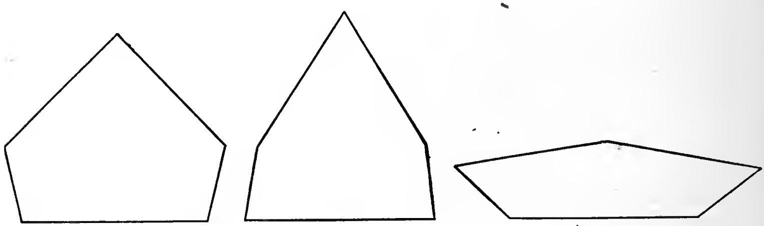
The word means "through the corners." Thus *AC* and *AD* are diagonals, but no diagonal can be drawn between *A* and *B*, since these corners are already connected by the edge *AB*.

41. How many diagonals can you draw from any one corner of a rectangle?
42. How many different diagonals can you draw between all the corners of a rectangle?
43. If you draw a diagonal of a square what is the shape of the parts into which you divide it?
44. Why cannot diagonals be drawn in a triangle?
45. Draw a polygon of five edges and then draw all the diagonals you can from any one corner.
46. Into how many parts have you divided the polygon?
47. What is the shape of the parts?

6. Polygons have names according to the number of the edges. The following is a list which you may use for reference without trying to commit the names to memory.

A pentagon has five edges; the word means "five corners," but every polygon has as many edges as it has corners.
A hexagon has six edges.
A heptagon, seven edges.
An octagon, eight edges.
A nonagon, nine edges.
A decagon, ten edges.
An undecagon, eleven edges.
A dodecagon, twelve edges.
A pentedecagon, fifteen edges.
An icosagon, twenty edges.

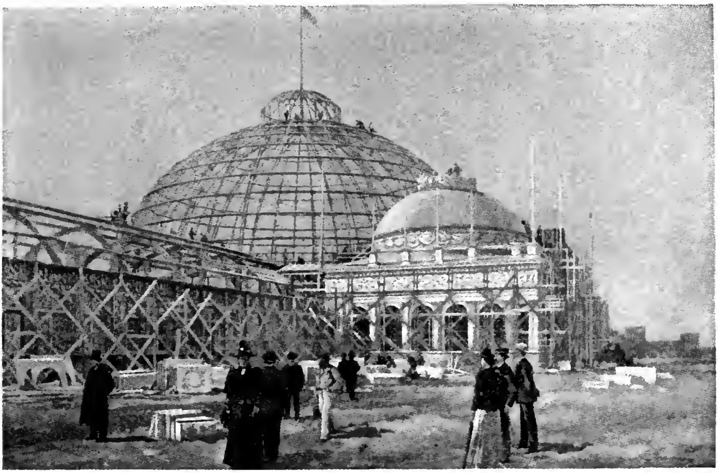
7. **Distortion of Polygons.** A polygon can have a countless variety of shapes without changing the lengths of its edges.



You can test this by taking small wooden rods to represent some polygon and fastening the ends together with pieces of tape which will act as hinges. You will find that whenever you pull or push the figure it will take a different shape.

Even a four-sided figure can be made to change its form; for a square can be turned into a rhombus, and a rectangle into a parallelogram.

Triangles are the only exception. When a triangle is once formed, you cannot alter its shape without disturbing the lengths of the edges.



Horticultural Building under Construction

Carpenters make an important use of these facts when they erect the frames of buildings or construct scaffolding.

The annexed picture represents a part of the Horticultural Building at the World's Fair in Chicago, as it appeared in the process of erection. The vertical and horizontal beams in the scaffolds form a series of rectangles, which might collapse under pressure even if the fastenings held firm. But you will notice that each rectangle has two boards nailed diagonally across, turning it into four triangles which from their shape add a geometric strength to that of the fastenings.

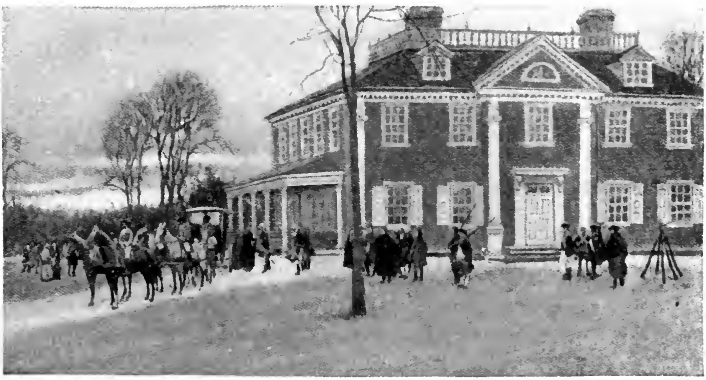
Another common example is a gate. This would be likely to "sag" after a time; that is, it would change from a square or a rectangle to a rhombus or a parallelogram; but the cross-bar extending from corner to

corner turns the quadrilateral into two triangles, which must keep their shape unless the wood decays or the joints become loose.

Let us see how many cross pieces are needed to hold a polygon to its shape. We have seen that a carpenter uses two for each rectangle in his scaffold and only one in a gate,

though both figures are quadrilaterals. But the carpenter has to consider the strength of materials, and a long beam is more likely to bend than a short one. Otherwise the question would simply be: how many diagonals must you draw in a polygon to divide it into triangles? Make experiments with polygons of various numbers of edges, choosing one corner from which to draw all the diagonals in each case. You will find that the necessary number is always three less than the number of the edges.

CHAPTER XII

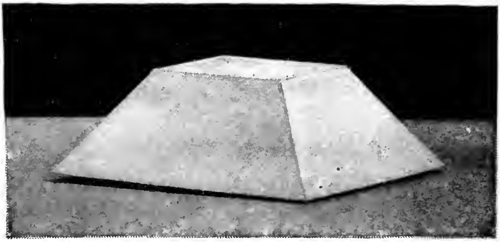


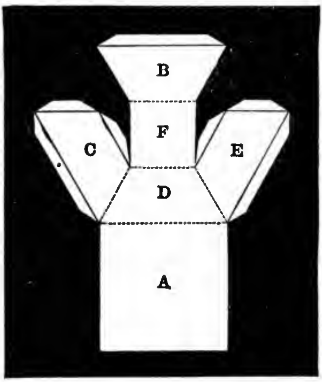
Washington's Headquarters at Cambridge

A FRUSTUM OF A PYRAMID

1. OBSERVE the roof in the picture of Washington's Headquarters. The sides slope upward from the eaves as if they were to meet at an apex and form the lateral faces of a pyramid; but, instead, they are cut short by the flat top of the roof, and we have only a *part* of a pyramid: such a part is called a *frustum*, a word meaning a "bit" or "piece" of anything. If a plane be passed through a pyramid parallel to its base, and the upper part (between the plane and the apex) be removed, the rest of the figure is called the *frustum of a pyramid.*

We will now make a model of the frustum of a square pyramid.





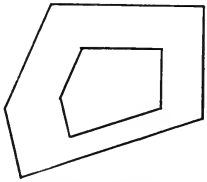
2. The diagram will need paper 14 cm. × 12 cm. (or $5\frac{1}{2}$ in. × 5 in.).

A is a square with edges 5 cm. (or 2 in.) long.

B, *C*, *D*, and *E* are equal trapezoids; one edge of each is 5 cm. (or 2 in.) long, and other edges are all 2 cm. 5 mm. (or 1 in.) long; the angles at the ends of the longer edges are all 60°.

F is a square with edges 2 cm. 5 mm. (or 1 in.) long.

3. The frustum has two bases. The lower base is the base of the pyramid itself, and may, therefore, have any number of edges and any shape. The upper base is formed by the cutting plane, and is an exact reduced copy of the lower base.



Similar Polygons

Two such polygons, which are shaped exactly alike, one being a reduced copy of the other, are called *similar* polygons.

The other faces of the frustum, that is, the lateral faces, are always trapezoids. They may or may not be equal or similar to one another.

1. What is the area of the lower base of the frustum which you have made?
2. What is the area of the upper base?
3. What is the shape of that part of the pyramid which is removed to form the frustum?
4. If a *prism* were cut by a plane parallel to the base, what would be the shapes of the parts into which the prism would be divided?

CHAPTER XIII

The Castle of Chillon

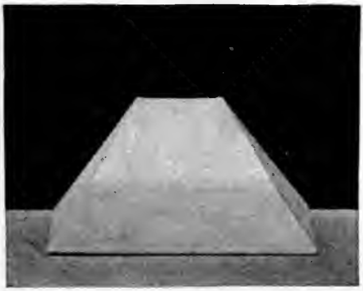
A TRUNCATED PYRAMID

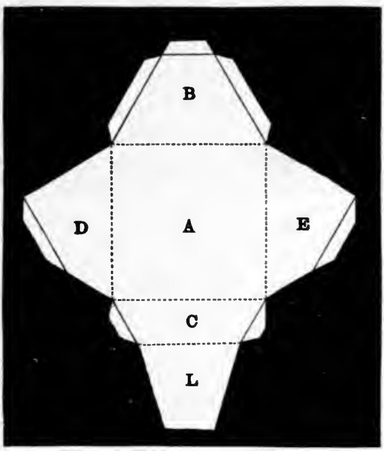
1. IF you examine the roofs of the two highest towers of the Castle of Chillon, you will see that while they are parts of pyramids they are not frustums, for the top of each tower is not a plane but an edge. The cutting plane, therefore, is not parallel to the base.

If a pyramid be cut by a plane not parallel to the base, and the part between this plane and the apex be removed, the rest of the figure is called a *truncated* pyramid.

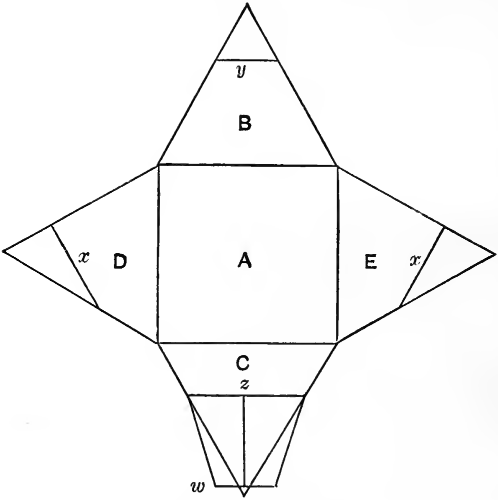
In the present instance the pyramids are cut by two planes, each extending to the base, and the result is a form which is common in architecture, being what is called "a hip roof."

We will now make a model of a truncated pyramid formed by a single cutting plane.





2. The diagram will need paper 16 cm. × 14 cm. (or $6\frac{1}{2}$ in. × $5\frac{1}{2}$ in.). The construction can be seen in the special figure.



A (see figure) is a square with edges 6 cm. (or 3 in.) long; upon each edge an equilateral triangle is drawn.

B is a trapezoid formed by marking off on the edges of one of the triangles distances of 4 cm. (or 2 in.) from the corners of the square, and drawing the fourth edge *y*, which will be 2 cm. (or 1 in.) long.

D and *E* are trapeziums formed by marking off on the edges of two opposite triangles distances of 4 cm. and 2 cm. (or 2 in. and 1 in.) from the corners of the square, and drawing the fourth edges *x*, which will meet one edge of each triangle perpendicularly.

C is a trapezoid formed by marking off on the edges of the last triangle distances of 2 cm. (or 1 in.) from the corners of the square, and drawing the fourth edge *z*, which will be 4 cm. (or 2 in.) long.

L is a trapezoid formed by drawing a perpendicular at the middle of *z*, 33 mm. (or $1\frac{5}{8}$ in.) long, and drawing the second edge *w* 2 cm. (or 1 in.) long, parallel to *z* and divided equally by the perpendicular; the other two edges can then be drawn and will each be equal to *x*.

3. The base of a truncated pyramid is the base of the pyramid itself, and may, therefore, have any number of edges and any shape.

The face formed by the cutting plane is called the *inclined section.*

The other faces, that is, the lateral faces, are quadrilaterals, either trapezoids or trapeziums.

Give the name of each face of your pyramid.
Are any of the faces equal?

CHAPTER XIV

Testing a Curved Surface

CURVED SURFACES AND LINES

1. WE will now begin the subject of curved surfaces and curved lines. Curved means "crooked," or "bent without corners." If you try to hold the straight edge of a ruler against some surfaces, you will find that you can do so only with certain positions of the ruler, sometimes with no position whatever; such surfaces are curved. Probably you can see objects about the room having curved surfaces, some of which

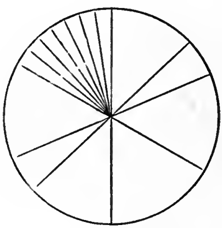
you can test with a ruler's edge. Perhaps you can find some curved surfaces upon which you can place the ruler's edge in certain directions, but not in all directions, as you can in the case of planes; but with most curved surfaces there will be no direction whatever in which you can hold the straight edge upon them.

In the case of water, if the amount is not great, the surface is considered to be a plane; but large bodies of water even when at rest must have curved surfaces; for these have the form of the earth's surface, which is a curve.

Curved edges, that is, curved lines, are formed by curved surfaces meeting other surfaces, curved or plane. Thus a plane surface can have a curved edge.

2. Of all faces bounded by curves the commonest is the *circle.*

Circle means "a ring."



Make a dot on paper; then, with the aid of a ruler draw from the dot a number of straight lines each two centimetres (or 1 in.) long. If you make these lines quite close together, you will see that their ends can be connected by a certain curved line; that line is called the *circumference* (cir-cum′-fer-ence) of the circle.

Circumference means "carrying around."

Each of the straight lines is called a *radius* (ra′-di-us) of the circle; radius means "a ray."

The point from which you drew the equal straight lines is called the *centre* of the circle.

The *circle* is the face itself, and is defined as a face bounded by a curved line all parts of which are equally distant from a point within called the centre.

The word "circle" is sometimes applied to the curve which bounds it, just as the word "ring" may refer to the curved boundary or to the enclosed space; but to be accurate you should call the curved line the circumference and the face itself the circle.

The radii measure the distance from the centre to the circumference and are therefore all equal to one another.

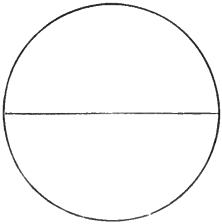
An *arc* is any part of a circumference; arc means "a bow."

An Arc

Knowing that the centre of a circle is equally distant from all parts of the circumference, can you decide from the picture of the archer whether or not his right hand is at the centre of the circle of which his bow forms an arc?

An Archer

A diameter (di-a′-me-ter) is a straight line drawn through the centre of a circle and bounded by the circumference. Every diameter divides the circle and the circumference into two equal parts.



A Diameter

How many radii does it take to make one diameter?
Are all the diameters of a circle equal to one another?

The circle and the rectangle are the commonest shapes in manufactured things. Probably you can see many objects about you which are made in these shapes; but in nature the circular is by far the commonest shape of all.

Field Artillery

In the picture of the cannon wheel, to what part of a circle does the tire correspond?

The spokes?

The axle?

Can you calculate the angle which each spoke makes with the one next to it?

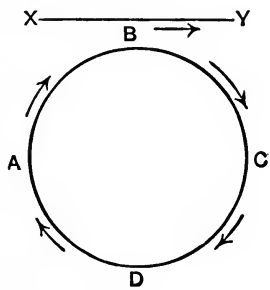
Are the spokes so arranged that each forms with another a diameter?

3. **Railroad Curves.** Curved lines can be parallel to one another; and then, as in the case of parallel straight lines, their distance apart remains unchanged.

The rails of a track are perhaps the most familiar instance of parallel curved lines.

Can you think of other instances?

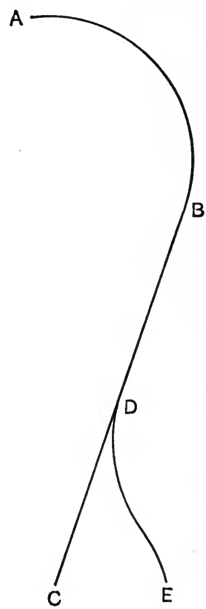
In one respect a curved line differs completely from a straight line. You have seen that a straight line *retains* one direction throughout its extent. Now a curved line *changes* its direction throughout its extent.



Thus in the annexed curve, if you place your pencil point at A and follow the curve around through B, C, and D, back to A, you will find that your pencil is always changing its direction. At C it is moving in the opposite direction to that in which it started; at D in the opposite direction to that in which it was moving at B; and finally it regains its first direction when it gets back to its starting-point at A.

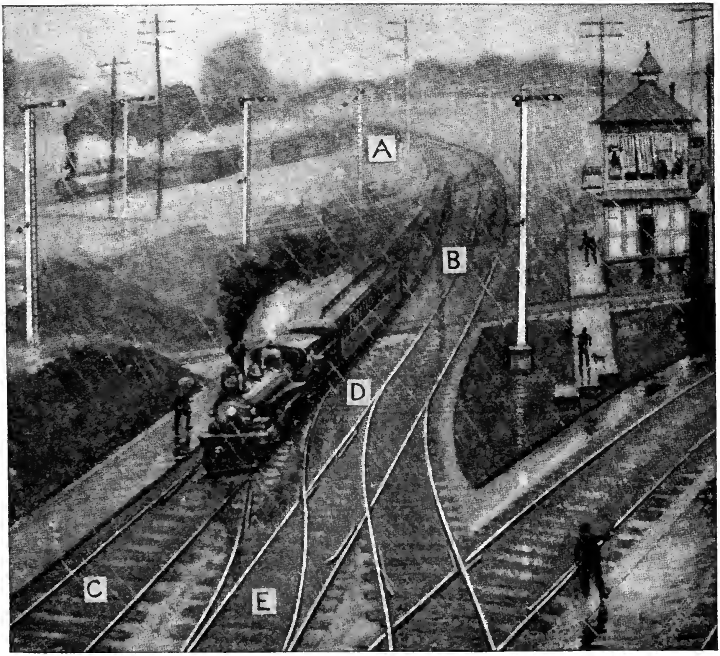
Notice, also, that if you compare the direction of the curve with the direction of some straight line, such as XY, there must have been a point where the curve and the straight

line had the same direction; but while the straight line continued in that direction, the curved line instantly changed to a new direction.



An important use of this truth is made in laying railroads so that trains may change their direction without running off the track. Suppose that the track AB is curved, and at B the direction is to be changed to a straight line. The men who are planning the route find the direction of the curve at B by drawing a radius to that point and making a perpendicular BC, along which they lay the track. So when a train from A reaches B, it keeps the direction it then has and goes in a straight line towards C. If at any point, as D, another curve has the same direction as BC, and both tracks are laid, towards E as well as towards C, a train from A on reaching D can go in either direction; but a "switch" prevents this by cutting off the track which is not to be used.

On the right of the picture you can see the switch-house from which the tracks are controlled.



Railroad Junction

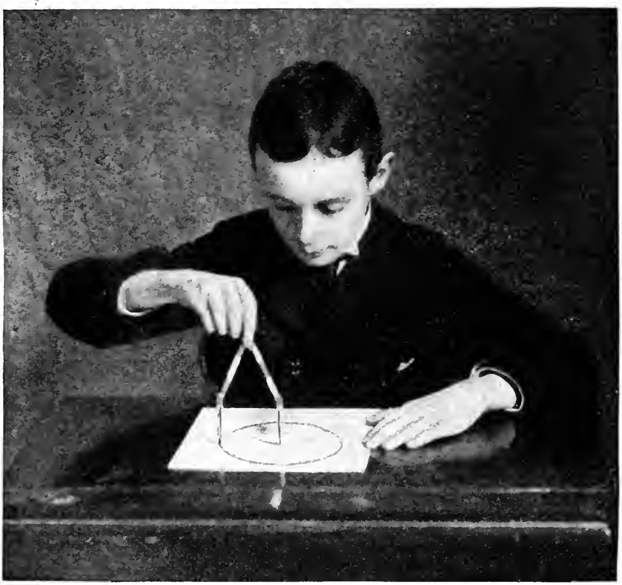
4. **Three Ways of drawing a Circumference.** Later on we shall have much more to say about circles; but now you need learn only enough to help you in drawing figures based upon circles.

First, we will see how circular faces can be marked off. There are many ways of doing this, three of which you may use.

First, a circumference can be drawn with an instrument called "a pair of compasses" or simply "compasses." A pair of compasses is a pair of dividers one of whose prongs carries at the end a pencil or a pen.

To use the compasses, place the pointed end firmly on the paper, and revolve the pencil end lightly until the line it makes comes around to its starting-point. The place where the fixed end rests is the centre of the circle; the distance between the points of the instrument is the length of the radius; and as that distance does not change during the revolution, the line you have made is the circumference of the circle, and the space enclosed is the circle itself.

With practice you will find that you can use the compasses best if you hold them between the thumb and forefinger, and press only firmly enough to keep the fixed end from sliding from its place at the centre.



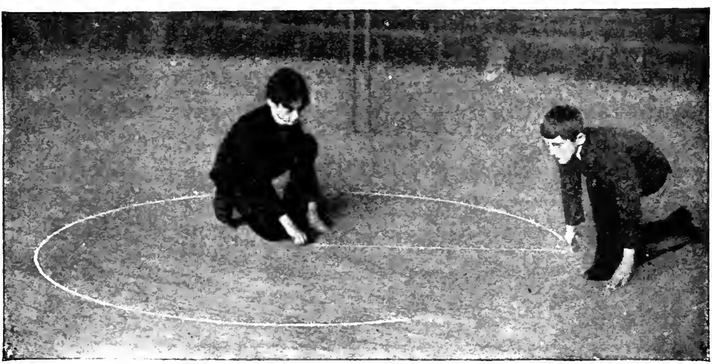
Drawing a Circle with a Pair of Compasses

Secondly, if you have no compasses, you can draw a circumference with the aid of a string which has a loop at each end. The length of the string will be the radius.

Place a pin in one loop where the centre of your circle is to be; then insert the pencil point in the other loop, draw the string tight against the paper, and move the pencil around. Its point will mark the circumference.

Drawing a Circle with a String

This method is a convenient one to use in case you wish to make a very great circle; for example, on the ground, where you would use a stake, a rope, and a pointed stick to mark the circumference.



Drawing a Circle with a Cord

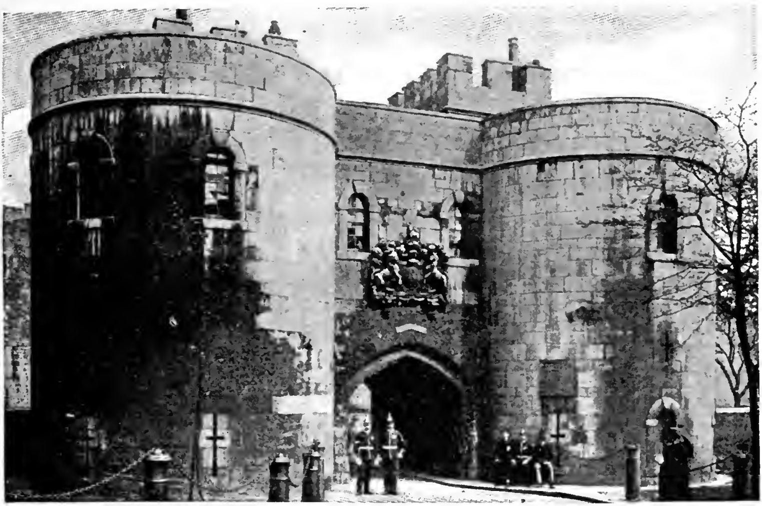
Thirdly, you can draw a circumference with the aid of a card, in which two small holes are made.

Drawing a Circle with a Card

The card is laid flat on the paper; through one of the holes a pin is inserted where the centre is to be; then the pencil point is inserted through the other hole, and the card is revolved with the pencil point marking the circumference as it moves.

This method has one convenience, that since the distance between the holes in the card is the length of the radius, a series of holes may be made at various distances noted on the card, thus avoiding constant measuring of radius lengths.

CHAPTER XV



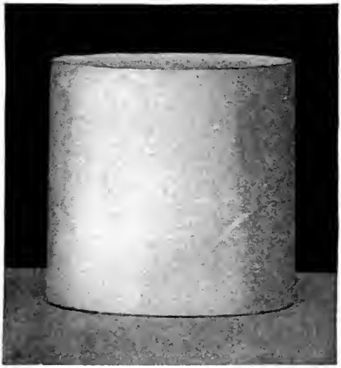
The Middle Tower

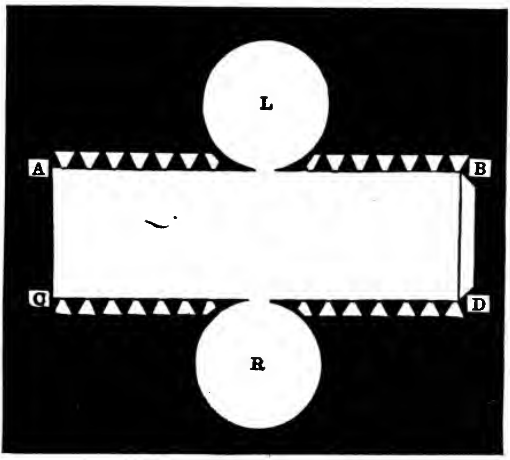
A CYLINDER

1. In the picture of "The Tower of London" you can see two examples of what is called "a round body."

This is a *cylinder* (cyl′-in-der), a word meaning "a roller." A cylinder has three surfaces, two of them being equal parallel planes called the bases of the cylinder, and the third being a curved surface. The edges of a cylinder are curved, and it has no corners. The cylinder is a common form in manufactured things; probably you can think of many objects which have this shape, such as pencils, parts of machines, etc.

We will now make a model of a cylinder.





2. The diagram will need paper 16 cm. × 15 cm. (or $6\frac{1}{2}$ in. × 6 in.).

First draw the rectangle *ABCD* with sides 15 cm. 7 mm. (or $6\frac{9}{32}$ in.) and 5 cm. (or 2 in.).

Then, with *L* and *R* as centres and a radius of 25 mm. (or 1 in.) draw circles so that they may just touch the longer sides of the rectangle. The lapels on these longer sides should be wedge-shaped and broader than usual. In cutting

out the figure, be careful not to separate the two circles entirely from the rectangle.

Paste first the edges *AC* and *BD*. Then paste the other lapels on the *outside* of the circular ends *L* and *R*; these edges should then be strengthened with a thin strip of paper; or you can make the circles a little smaller so that they may fit inside the figure, and then cover them with circles of full size.

3. **1.** How does this cylinder resemble a prism?
2. What is the smallest number of faces which can bound a prism? What shape has its base?
3. What is a straight line? Are there various kinds of straight lines?
4. What is a curved line? Are there various kinds?
5. What is a circumference?
6. What is a circle?
7. What is an arc?
8. What is the centre of a circle?
9. What is the difference between a circle and a circumference?
10. What is the difference between a diameter and a radius?
11. What was the shape of the quadrilateral which you bent so as to form the curved surface of the cylinder?
12. Which two edges of the quadrilateral join the curved edge of the bases?
13. How do those edges compare in length with the circumference of the bases?
14. Which two edges of the quadrilateral are equal to the distance between the bases of the cylinder?
15. This quadrilateral has formed a curved surface: what is a curved surface?
16. How can you test whether a surface is curved or not?
17. Can a straight line be drawn on the curved surface of a cylinder?
18. Can several straight lines be so drawn? If so, how do their directions compare with each other?
19. Can you imagine your cylinder to be fitted exactly into a cubical box? If so, what would be the dimensions of the interior of the box?
20. Can you hold a ruler with its edge against the curved surface of your cylinder in such a position as will show that a straight line could *not* be drawn in that direction on the surface?

4. **The length of the circumference of any circle** is *about* three times the length of its own diameter: it is really a little more than three times; three and one-seventh times would be more exact.

You can test this in two ways. First you refer back to the diagram from which you made the cylinder: —

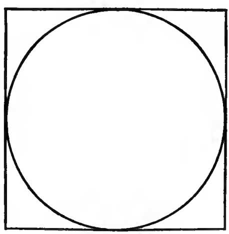
21. What is the length of the diameter of one of the circles?
22. What is the length of the edge of the rectangle which is to be bent around so as to meet the circumference of the circle?

23. How many times do you find one length to be contained in the other?

Secondly, you can make measurements on the surface of the completed cylinder by bending a tape measure, or a narrow strip of paper, around the curved surface close to the base.

24. About what is the length of a circumference whose diameter is 7 cm. long?

5. **The area of a circle** is *about* three-fourths of the area of a square which has an edge equal to a diameter of the circle. Thus in the annexed figure the circle occupies about three-fourths of the square; and the parts which lie outside the circle, at the corners of the square, together make about one-fourth of the square.



We will test this by an experiment, and at the same time find the volume of the cylinder.

Make another cylinder, leaving out one of the bases, and take the cube you have used for measuring.

First place the bases of the two figures together, and see that the diameter of the cylinder is equal to the edge of the square.

Then place the two figures on a horizontal plane, and with the aid of a ruler resting across their tops see that their heights are equal.

Then experiment by filling them with sand or water, etc. You will find that it takes four fillings of the cylinder to make three of the cube; or if you fill the cylinder once, and pour the contents into the cube, the level inside the cube will be three-fourths of the whole height.

Now, since the two figures have the same height, the difference in their volumes depends on the difference in the areas of their bases. So the circular base of the cylinder is three-fourths of the square base of the cube.

25. What is the length of an edge of your cube?
26. What is the length of the diameter of the base of your cylinder?
27. What is the area of the base of your cube?
28. What is the area of the base of your cylinder?
29. What is the volume of your cube?
30. What is the volume of your cylinder?
31. What is the area of the rectangle which was bent to form the curved surface of your cylinder?
32. What, then, is the area of the curved (or lateral) surface of your cylinder?
33. How would you find the lateral surface of a cylinder given to you ready made?
34. If you knew the area of the base of a cylinder and the height, how would you find the volume?
35. What is the volume of a cylinder whose height is 8 cm., and the area of whose base is 20 sq. cm.?
36. What is the volume of the greatest cylinder which can be placed in a cubical box 6 inches deep?
37. How many square inches are there in the entire surface of a cylinder whose lateral surface is formed from a rectangle 5 in. long and 4 inches wide, and whose bases are circles?
38. What is the volume of the above cylinder?

The length of a circumference is about three (more nearly $3\frac{1}{7}$*) times the length of its diameter.*

The area of a circle is about three-fourths of the area of the square on its diameter.

The area of the lateral surface of a cylinder is the product of the circumference of its base and the distance on the surface between the bases.

The volume of a cylinder is the product of the area of its base and altitude.

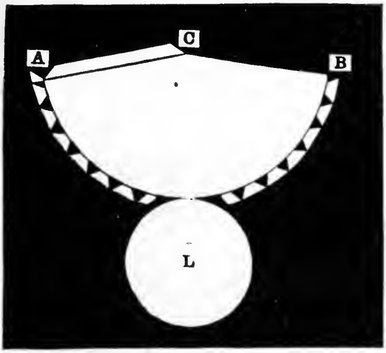
CHAPTER XVI

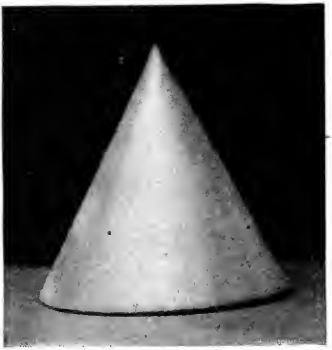
Mount Fuji

A CONE

1. In the picture of Mount Fuji you have another example of a round body. This is a *cone*, a word which originally meant "a peak," that is, the top of a mountain. A cone has two surfaces, one plane and the other curved. The plane surface is the base of the cone, and is bounded by a curved line. The curved surface begins at a point called the apex or vertex of the cone, and extends to the base.

We will now make a model of a cone.





2. The diagram will need paper 12 cm. × 11 cm. (or 5 in. × $4\frac{1}{2}$ in.).

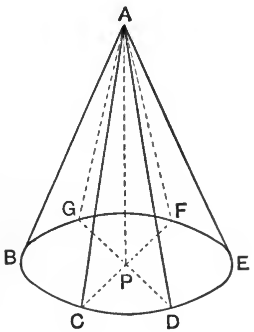
First draw the angle *ACB*, 160°.

Then with the vertex *C* as a centre, and a radius of 5 cm. 6 mm. (or $2\frac{1}{4}$ in.) draw the arc *AB* between the sides of the angle.

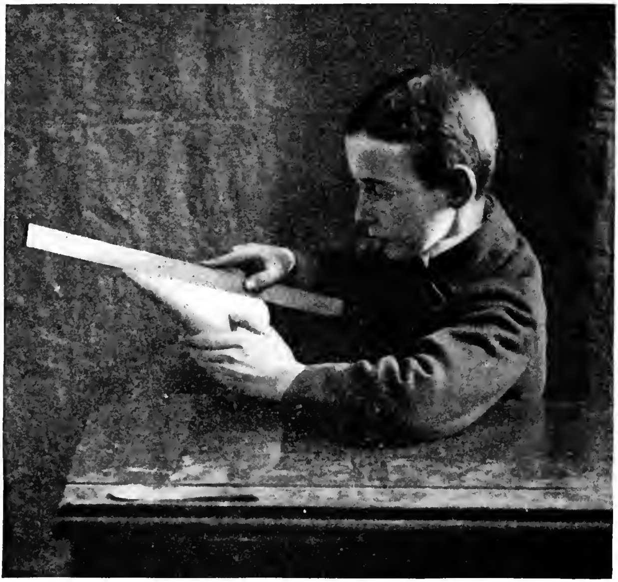
Then with *L* as a centre and a radius of 25 mm. (or 1 in.) draw a circle just touching the arc.

Make wedge-shaped lapels on the arc, and be careful not to cut the circles entirely apart. The lapels should be pasted on the outside of the circular base, and the edge should be strengthened with a thin strip of paper or a second circle, as you did in the case of the cylinder.

3. The plane figure which you bent so as to form the curved surface of the cone is called a *sector;* it is a part of a circle bounded by two radii and an arc.



The *height* of a cone is the perpendicular distance from the apex to the base, as AP (see figure). In the cone you have made, this line passes through the centre of the base.



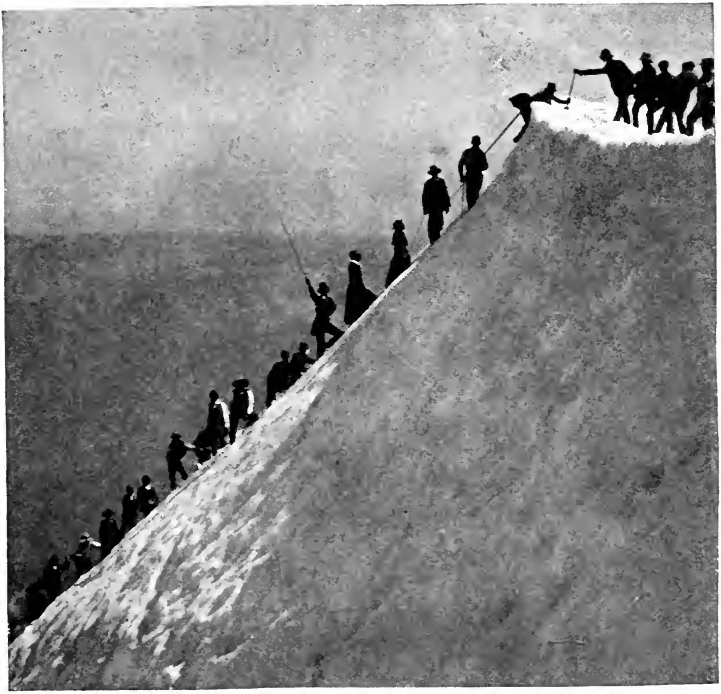
Testing the Surface of a Cone

The *slant* height of a cone is the distance from the apex to the circumference of the base, as AB, AC, AD, etc. (see figure); it is measured in a straight line, and is the only possible case of straight lines on the curved surface of the cone, as you can see by holding a ruler's edge against the surface. In your cone the slant heights are all equal.

In the picture of "Cloud Cap Mountain" we have an example of a part of a cone called a frustum. The frustum of a

cone is the part which lies between the base and a plane which cuts the cone parallel to the base.

The part cut off above the plane is a smaller cone.



Cloud Cap Mountain

4. **The area of the curved (lateral) surface** of your cone is equal to the length of the circumference of the base multiplied by one-half the slant height.

First, we will find the length of the circumference by calculation and test the answer by measurement: —

1. What is the length of the diameter of the base?
2. By what will you multiply the diameter in order to find the length of the circumference?
3. What, then, is the length of the circumference?

Now measure the length of the circumference with a tape or narrow strip of paper, and see if the two results agree.

Next, we will find the slant height from the diagram from which you made the cone, and test the answer by measurement:—

4. What line on the diagram corresponds to the slant height?
5. What is its length?

Now measure the slant height on the surface of the cone, beginning at the apex. Remember that we wish to measure a *straight* line, although on a curved surface; and the only possible straight lines on the lateral surface of a cone are those which pass through the apex or would do so if prolonged.

Lastly, we will find the area of the lateral surface by multiplying the length of the circumference by one-half the slant height. The answer is about 44 sq. centimetres, or, if you have used English measurements, about 7 sq. inches.

A Medicine-Man's Lodge

5. **The volume of a cone** is equal to one-third of the volume of a cylinder whose base and height are equal to those of the cone.

We will test this by an experiment. Make another cone, leaving out the base, and take the cylinder you have used for measuring.

First, place the bases of the two figures together and see that they are equal. Then with the aid of a ruler resting across their tops see that their heights are equal. Then compare the volumes of the figures by filling them with sand or water, etc. You will find that it takes three fillings of the cone to make one of the cylinder.

6. Now the volume of a cylinder is equal to the area of its base multiplied by its height: what then is the volume of your cone?

7. The picture of "the medicine-man's lodge" represents a cone-shaped tent with a diameter and height each about 15 feet. The length of the poles from the top to the outer edge is about 17 feet.

How many square feet of material was needed to make this lodge?

How many cubic feet does it contain?

8. What is the volume of a cone whose height is 6 cm., and the area of whose base is 20 sq. cm.?

9. What is the volume of a cone whose height is 12 inches, and the diameter of whose base is 8 inches?

10. If a cone and a cylinder have equal bases, but the cone is three times as tall as the cylinder, how do their volumes compare?

The area of the lateral surface of a cone is one-half the product of its circumference and slant height.

$$\textit{Lateral surface cone} = \frac{\textit{circumference of base} \times \textit{slant height}}{2}$$

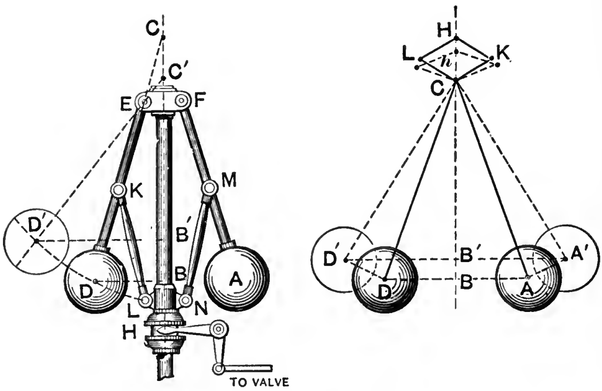
$$= \frac{\textit{circumference of base}}{2} \times \textit{slant height}$$

$$= \textit{circumference of base} \times \frac{\textit{slant height}}{2}$$

The volume of a cone is one-third the product of the area of its base and altitude.

$$\textit{Volume cone} = \frac{\textit{base} \times \textit{altitude}}{3} = \frac{\textit{base}}{3} \times \textit{altitude} = \textit{base} \times \frac{\textit{altitude}}{3}$$

CHAPTER XVII



Watt's Governor

SOLIDS OF REVOLUTION. — THE SPHERE

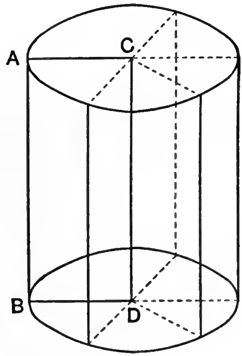
1. JUST as a flame at the end of a stick which is whirled rapidly around looks like a continuous circle of fire, so various plane figures when revolved about an axis give the appearance of solid bodies.

Thus in "Watt's Governor" the triangle formed by the two rods of the governor which carry the balls looks like a cone when the engine is in motion, and the hexagon *EFMNLK* looks like two frustums of cones with their larger bases together.

Hence certain solid figures are called "solids of revolution;" for we can imagine them as formed or generated by the revolution of plane figures. There are three principal

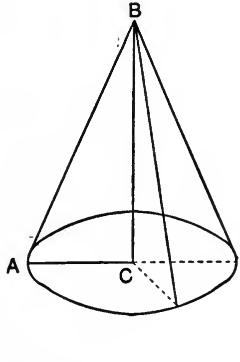
solids of revolution, two of which—the cylinder and the cone—you have already studied.

A cylinder is generated by the revolution of a rectangle about one of its edges.

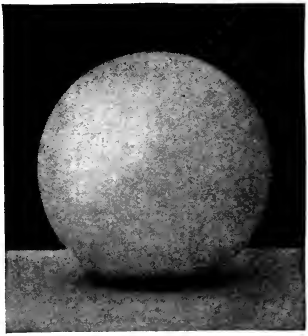


Thus the rectangle *ABCD* revolving about *CD* as an axis forms a cylinder whose height is *CD*, and whose bases are circles with radii equal to *BD*.

A cone is generated by the revolution of a right triangle about one of the sides of the right angle. Thus the triangle *ACB*, revolving about *BC* as an axis, forms a cone whose height is *BC*, slant height *AB*, and whose base is a circle with a radius equal to *CA*.



We will now consider the third solid of revolution. If you spin a coin on its edge, you will see that it takes the appearance of a solid figure different from those which we have hitherto studied. The coin is a circle which has revolved about one of its diameters; but the same effect would be produced if only a semi-circumference were to revolve on its diameter.



2. This figure is called a sphere; it is a very common shape in nature, manufactures, and the arts.

The word is derived from a Greek word meaning "a ball, or globe."

The *surface* of a sphere is curved, and all parts of it are equally distant from a point within the sphere, which is the centre.

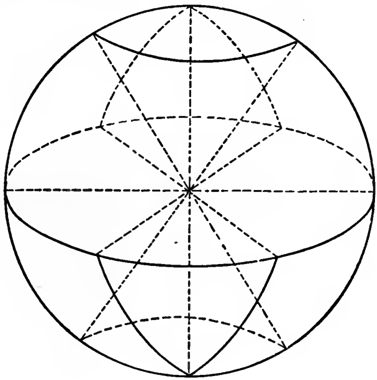
A *radius* of a sphere is a straight line drawn from the centre to the surface.

A *diameter* of a sphere is a straight line drawn through the centre and bounded at each end by the surface. Thus a diameter is equal to two radii.

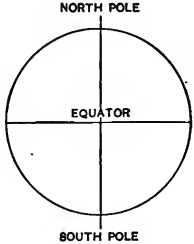
All the radii of a sphere are equal, and all the diameters are equal.

Poles of a sphere are the ends of any diameter, and are therefore points.

The word, as used in geometry, is derived from a Latin word meaning "a pivot." Thus the poles of the earth are the two points at the ends of the diameter on which as an axis the earth revolves.



No straight line can be drawn on the surface of a sphere, as you can easily see by trying to hold a ruler's edge against the surface. Circumferences of circles, however, can be drawn; and the circles are of two kinds, — great circles and small circles.

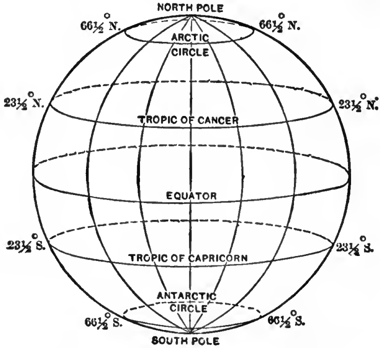


SPHERE

Poles and Axis

A *great circle* has the same radius and diameter as the sphere itself; it is the greatest possible circle whose circumference can be drawn on the surface of the sphere.

The equator and meridians of longitude are examples of great circles of the earth, the meridians being semicircles.



Parallels and Meridians

A *small circle* is a circle whose radius is less than the radius of the sphere.

Parallels of latitude are examples of small circles of the earth.

Every great circle divides the sphere into two equal parts called *hemispheres;* the word means "half a sphere." The hemisphere is a common shape in the domes of buildings.

In the picture of Jerusalem you can see two hemispherical domes.
On the dome of the Greek Church how many great circles are indicated?
How many small circles?
Of which kind is the circle which bounds the base of this dome?
On the dome of the Church of the Holy Sepulchre of which kind are the circles indicated?

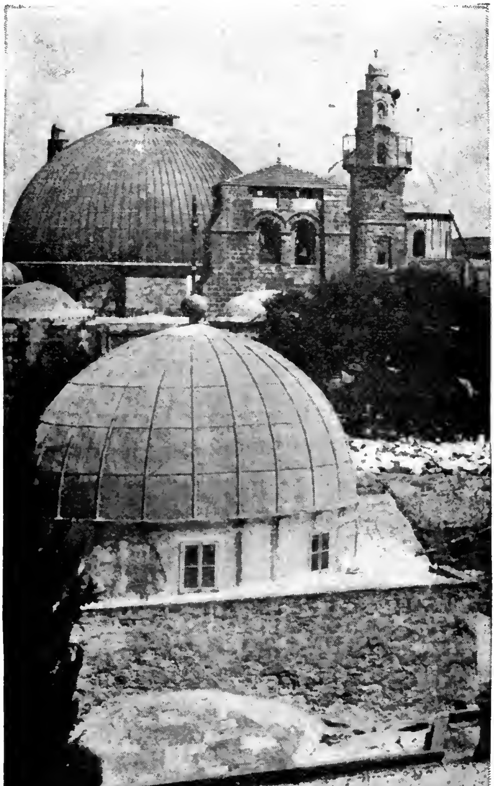
Zones are portions of the surface bounded by the circumferences of parallel circles.

The word "zone" is derived from a Greek word meaning "a belt."

The circumferences which bound the zone are called the *bases* of the zone.

The Torrid and Temperate Zones on the earth's surface are examples of zones of two bases. The bases of the Torrid Zone are the Tropic of Cancer

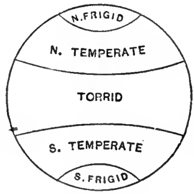
and the Tropic of Capricorn. The Frigid Zones are examples of zones of one base. The one base of the North Frigid Zone is the Arctic Circle; but we can imagine the other base to lie outside the earth, at the North Pole.



Churches in Jerusalem

3. **The area of the surface of a sphere** is exactly equal to the area of four great circles.

Thus if the diameter is 5 cm., the area of a great circle is about $19\frac{1}{2}$ sq. cm., and the area of the sphere is about 78 sq. cm.



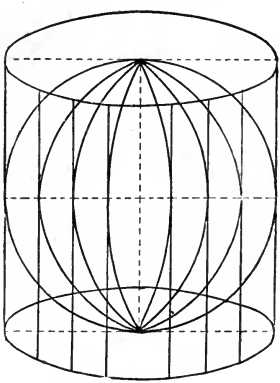
ZONES

The surface of a sphere is also exactly equal to the lateral (curved) surface of a cylinder which will just contain the sphere.

An important use is made of this truth, in drawing maps so as to represent the curved surface of the earth by a flat map on which the parallels and meridians appear as straight lines.

The map is drawn on the lateral surface of a cylinder which is then unrolled so as to form a rectangle, thus reversing the process by which you made your cylinder.

This is called "drawing maps by Mercator's projection."



4. **Volume of a Sphere.** If we consider a circumference to be three times as long as its diameter, the volume of a

sphere can be calculated by multiplying the diameter by itself, multiplying again by the diameter, and then dividing by 2.

Thus, if the diameter of a sphere is 5 cm. long, the volume of the sphere is $5 \times 5 \times 5 \div 2$, or $62\frac{1}{2}$ cubic centimetres.

Find the areas of the surfaces of the following spheres:—

1. Diameter 4 cm.
2. " 6 "
3. " 8 inches.
4. Radius 4 cm.
5. " 6 inches.

Find the volumes of the following spheres:—

6. Diameter 2 cm.
7. " 3 "
8. " 4 inches.
9. Radius 1 cm.
10. " 2 inches.

The area of the surface of a sphere is four times the area of a great circle.

$$\textit{The volume of a sphere} = \frac{\textit{diameter} \times \textit{diameter} \times \textit{diameter}}{2}$$

CHAPTER XVIII

FIGURES FOR PRACTICE

THE general name for a figure of three dimensions is *polyedron*, which means "having many faces."

The following figures are more difficult to construct and observe. Many of them are combinations or parts of figures which you have already made. Some resemble crystal forms which occur in nature.

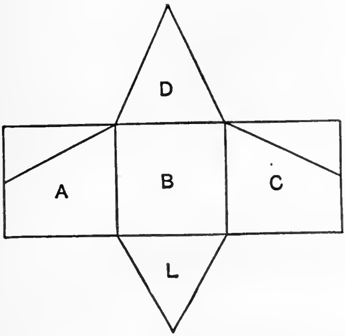
Three are *regular* polyedrons; that is, their faces are equal regular polygons, and their diedral angles are equal.

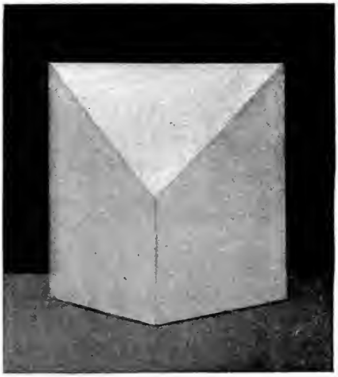
Only five regular polyedrons are possible, of which you have already constructed two, the cube and the equilateral triangular pyramid.

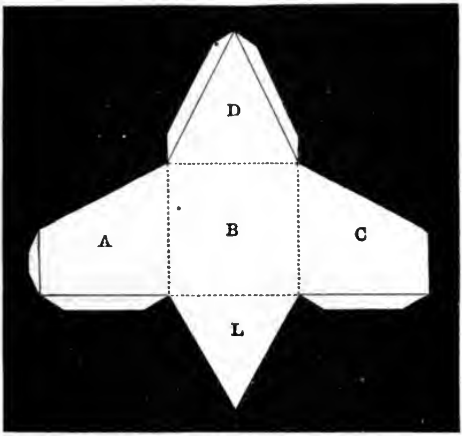
When you have completed a figure, you should examine it carefully, and see if you can answer the following questions:

1. Is the figure a combination of smaller ones? If so, what are their shapes?
2. Is the figure a part of another? If so, what is the shape of the other? How was the division made?
3. How many faces has the figure?
4. Describe the shapes of the faces; and if there are several kinds, find the number of each kind.
5. How many edges has the figure?
6. What are the lengths of the edges?
7. How many corners or solid angles has the figure?
8. How many faces form the solid angles?
9. How many diedral angles has the figure?
10. What are the sizes of the diedral angles?
11. How many line angles are there on the surface of the figure?
12. What are the sizes of the line angles?
13. What is the volume of the figure?

The volume should be found by experiment. Before closing the last face fill the figure with sand, and pour the contents into a cube, where measurements can easily be made.







A TRUNCATED TRIANGULAR PRISM

The diagram will need paper 16 cm. × 15 cm. (or $6\frac{1}{2}$ in. × 6 in.).

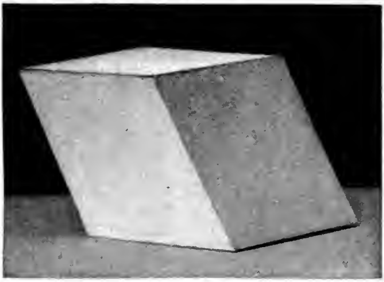
The construction can be seen from the special figure.

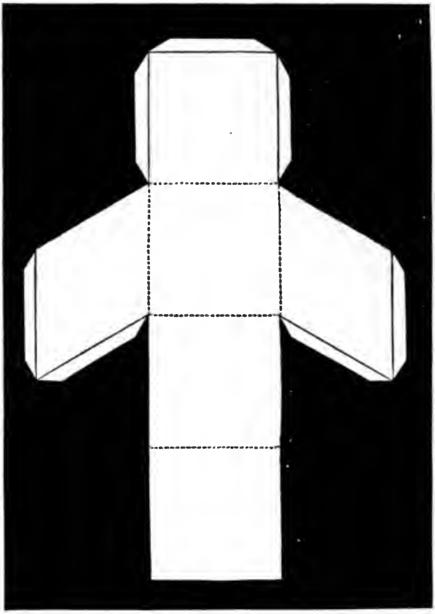
A, *B*, and *C* are equal squares with edges 5 cm. (or 2 in.) long.

From the upper corners of *B* lines are drawn to the middle points of the outer edges of *A* and *C*.

With two edges equal to these lines an isosceles triangle *D* is constructed on the upper edge of *B*.

L is an equilateral triangle constructed on the lower edge of *B*.

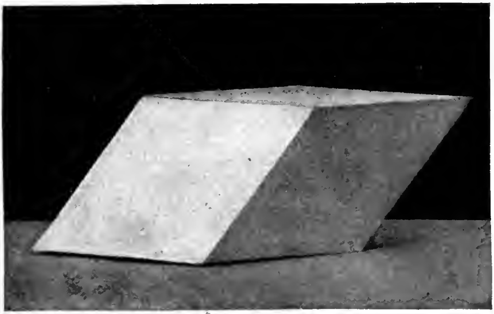


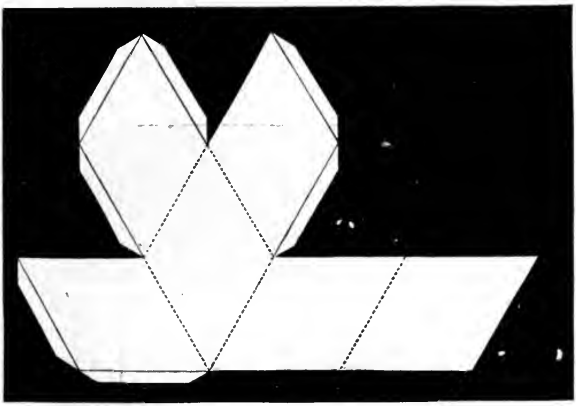


A QUADRANGULAR PRISM

The diagram will need paper 20 cm. 5 mm. × 15 cm. (or $8\frac{1}{4}$ in. × 6 in.).

The surface consists of four equal squares with edges 5 cm. (or 2 in.) long, and two equal rhombuses with angles 60° and 120° and edges 5 cm. (or 2 in.) long.





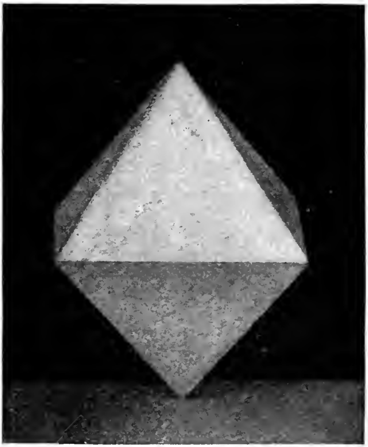
A RHOMBIC PRISM

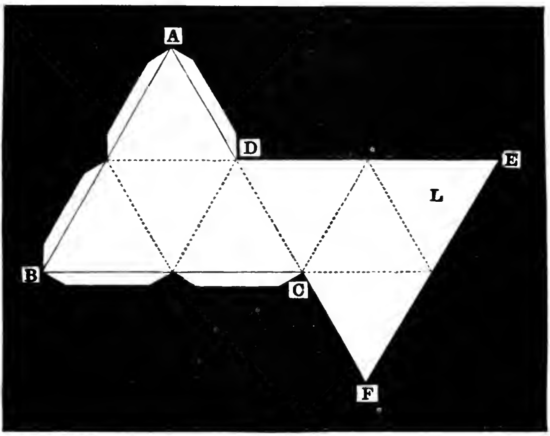
The diagram will need paper 20 cm. × 14 cm. (or 8 in. × 6 in.).

The surface consists of equal rhombuses with angles 60° and 120° and edges 5 cm. (or 2 in.) long.

These prisms are to be used for comparison with the cube. How do the three figures compare :—

1. In the number of edges?
2. In the total length of edges?
3. In the number of faces?
4. In the total area of their surfaces?
5. In their volumes?



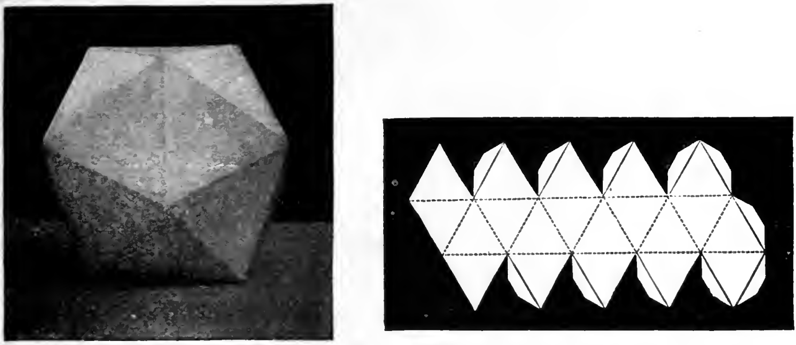


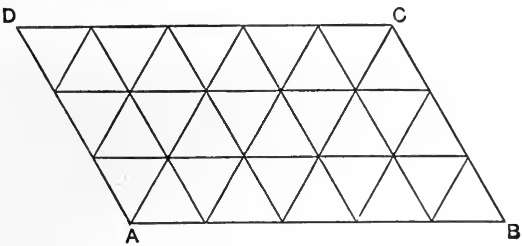
THE REGULAR OCTAEDRON

The diagram will need paper 18 cm. × 14 cm. (or $7\frac{1}{2}$ in. × 6 in.).

ABC and *DEF* are equilateral triangles with edges 1 dcm. (or 4 in.) long; *D* is the middle point of *AC*. These two triangles are each divided into four equilateral triangles by joining the middle points of the edges.

This figure, the third in the series of regular polyedrons, is called an octaedron, because it has eight faces.





THE REGULAR ICOSAEDRON

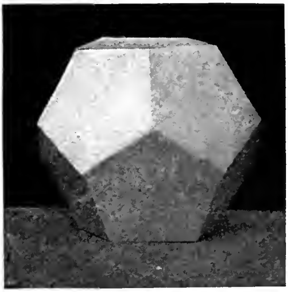
The diagram will need paper 17 cm. × 8 cm. (or $7\frac{1}{2}$ in. × 3 in.).

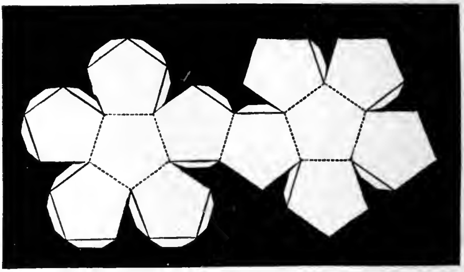
The construction can be seen from the special figure.

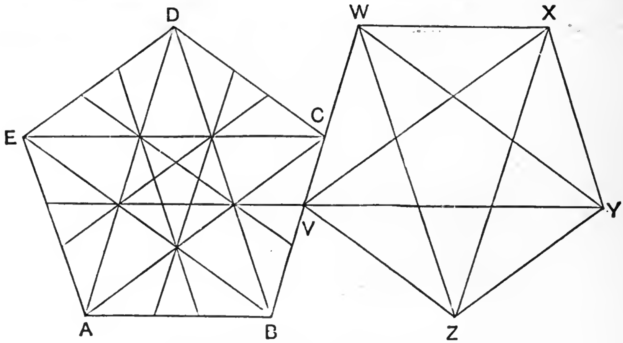
ABCD is a parallelogram with angles 60° and 120°, and edges 12 cm. 5 mm. and 7 cm. 5 mm. (or 5 in. and 3 in.) long.

Each edge is divided into equal parts 2 cm. 5 mm. (or 1 in.) long. Then the points of division are joined by three series of parallel lines as in the figure, thus dividing the parallelogram into thirty equilateral triangles, of which ten — having a side on the top or bottom of the parallelogram — are afterwards removed, leaving only enough to form lapels of the triangles which remain.

The above figure, the fourth of the series of regular polyedrons, is called an *icosaedron* (i-cos-a-e′-dron) because it has twenty faces.







THE REGULAR DODECAEDRON

The diagram will need paper 17 cm. × 9 cm. (or 7 in. × 4 in.).

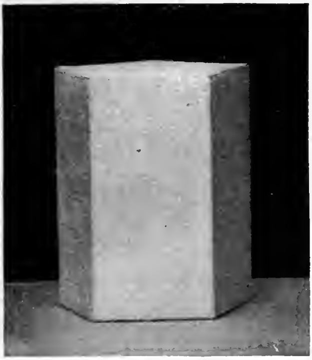
The construction can be seen from the special figure.

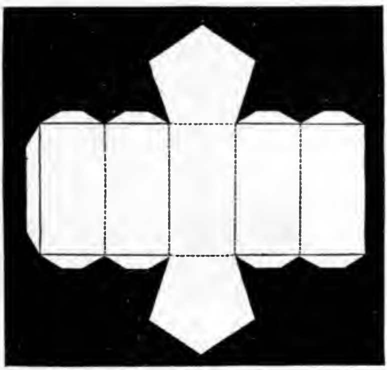
ABCDE is a regular pentagon, each angle being 108°, and each edge 5 cm. (or 2 in.) long.

The five diagonals *AC*, *AD*, etc., are drawn, forming a smaller pentagon within the first. Then all the diagonals of the smaller pentagon are drawn and prolonged to the edges of the larger one, thus forming five more pentagons.

Next, the regular pentagon *VWXYZ* is constructed, *V* being a vertex of one of the smaller pentagons, and *VW* one of the edges prolonged so that *VW* may be equal to *BC*. The diagonals are drawn as before.

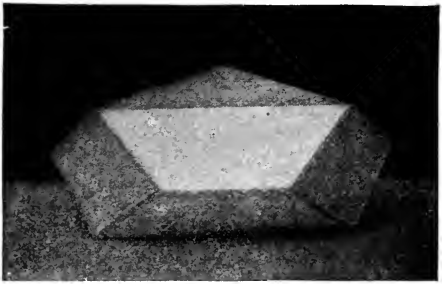
This figure, the fifth and last of the series of regular polyedrons, is called a dodecaedron, because it has twelve faces.

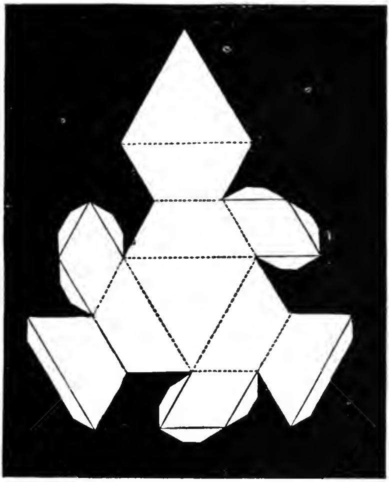


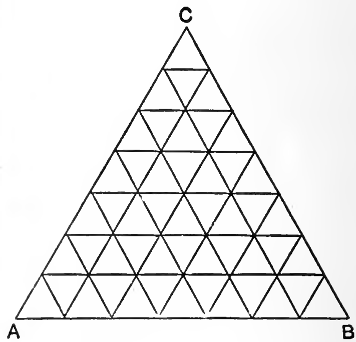


A PENTAGONAL PRISM

The diagram will need paper 13 cm. × 12 cm. 5 mm. (or $5\frac{1}{4}$ in. × 5 in.).
The faces consist of rectangles and regular pentagons.
The rectangles have edges 5 cm. and 2 cm. 5 mm. (or 2 in. and 1 in.) long.
The pentagons have edges 2 cm. 5 mm. (or 1 in.) long, and angles of 108°







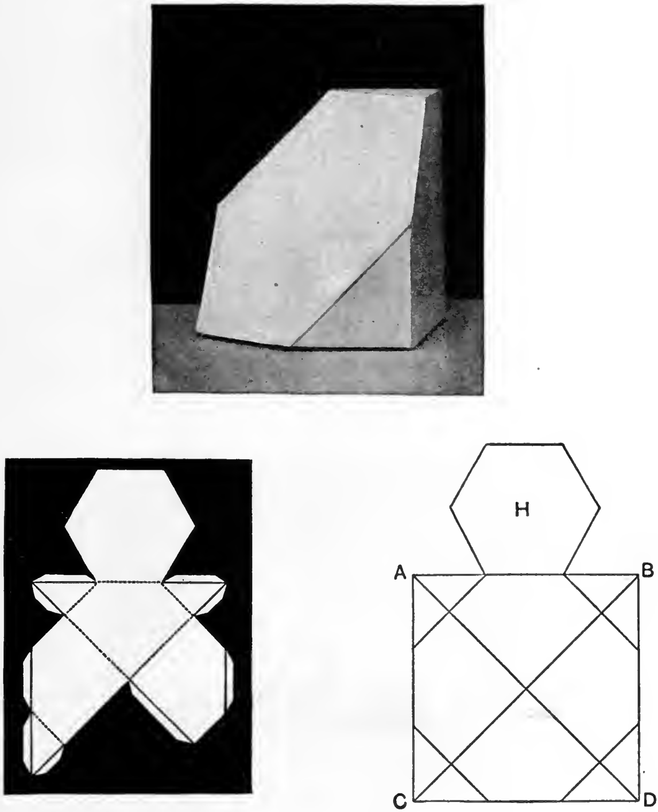
CRYSTAL OF SPINEL

The diagram will need paper 18 cm. × 16 cm. (or $7\frac{1}{2}$ in. × $6\frac{1}{2}$ in.).

ABC is an equilateral triangle with edges 17 cm. 5 mm. (or 7 in.) long. Each edge is divided into seven equal parts 2 cm. 5 mm. (or 1 in.) long, and lines are drawn parallel to the edges of the triangle, connecting the points of division, thus forming smaller equilateral triangles. The lines which appear in the diagram lie along the lines which are in the special figure.

The faces consist of equilateral triangles with edges 5 cm. (or 2 in.) long; rhombuses with angles 60° and 120°, and edges 2 cm. 5 mm. (or 1 in.) long; and trapezoids with angles 60° and 120°, and edges 5 cm. and 2 cm. 5 mm. (or 2 in. and 1 in.) long.

This model resembles the crystal of spinel.



CRYSTAL OF COPPER

The diagram will need paper 12 cm. × 8 cm. (or 5 in. × $3\frac{1}{2}$ in.).

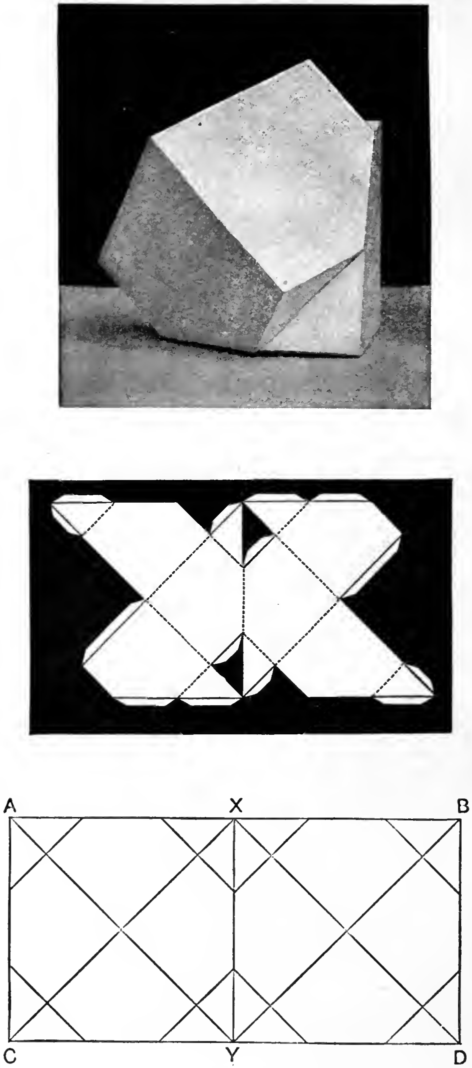
The construction can be seen from the special figure.

ABCD is a square with edges 7 cm. 5 mm. (or 3 in.) long.

The edges are divided each into three equal parts 25 mm. (or 1 in.) long; and parallel lines are drawn connecting the corners and other corresponding points of division.

H is a regular hexagon constructed on the middle division of the upper edge of the square.

This model resembles one of the crystal forms of copper.



A TWIN CRYSTAL OF CALCITE

The diagram will need paper 16 cm. × 8 cm. (or $6\frac{1}{2}$ in. × $3\frac{1}{2}$ in.).

The construction can be seen from the special figure.

ABCD is a rectangle with edges 15 cm. and 7 cm. 5 mm. (or 6 in. and 3 in.), divided into two squares by the line *XY*.

The edges of the squares are divided each into three equal parts 25 mm. (or 1 in.) long; and parallel lines are drawn connecting the corners and other corresponding points of division.

This model resembles a crystal form of calcite, called "a twin-crystal," as it consists of two interpenetrating cubes.

PART II

POINTS, LINES, ANGLES, POLYGONS, AND CIRCLES

CONSTRUCTIONS, MENSURATION, SIMILAR FIGURES, AND SURVEYING

CHAPTER XIX

POINTS AND LINES

I. ARRANGEMENTS of points with reference to each other in the same straight line.

1. In how many different orders can two points be placed with reference to each other in the same straight line?

Let *a* and *b* be two points.

1st. *a* can be placed before *b*, as 

2nd. *b* can be placed before *a*, as 

Therefore there are two different arrangements.

2. In how many different orders can three points be arranged in one straight line?

Let *a*, *b*, and *c* be the three points.

You have seen by the preceding problem that two points, *a* and *b*, can be arranged in two different orders

Now taking the first group 

, notice that *c* can be introduced into it in three different ways,

Likewise in the second group, 

, *c* can be inserted in three different ways,

Therefore there are six different arrangements possible.

3. In how many different orders can four points be arranged in one straight line?

Take one of the groups of three points, and introduce the fourth point into it in the various positions; then do the same with each of the other groups of three points. You will find that there are twenty-four possible orders in all.

4. In how many different orders can five points be arranged in one straight line?

Write out only one set of groups, but calculate the total number.

5. Find the number of orders for six points.

By examining the method which you have used in the preceding problems, you can obtain a rule by which you can calculate the total groups which any number of points would form.

2 points	give	2	groups	=	1×2
3 "	"	6	"	=	$1 \times 2 \times 3$
4 "	"	24	"	=	$1 \times 2 \times 3 \times 4$.

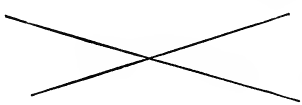
Therefore, to calculate the total groups for any number of points, multiply together the numbers from 1 up to and including the number of points.

6. Find by calculation the total number of arrangements of 7 points in the same straight line.
7. What series of numbers, if multiplied together, would give the total number of groups for 10 points?

2. Points determined by intersecting straight lines.

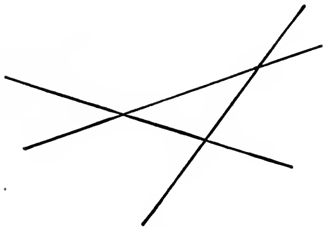
1. In how many points can two straight lines cut each other?

Two straight lines can cut each other at only one point.



2. In how many points can three straight lines cut each other?

Two straight lines can cut each other at one point, and a third straight line can cut the other two each at one point; therefore three straight lines can cut each other at three points.



3. According to the preceding problem, three is the *greatest* number of points in which three straight lines can cut each other: can you draw three straight lines so that they can cut each other in only two points?
4. Can you draw the three lines so that they will cut each other in only one point?
5. Can you draw them so that they will not cut each other in any point?
6. What is the greatest number of points in which four straight lines can cut each other? Draw a diagram.
7. Five straight lines? Draw a diagram. [Ans. 10 points.]
8. Six straight lines? " " "

From the preceding problems you can obtain a rule by which you can calculate the greatest number of points of intersection which any number of straight lines can have.

2	straight lines	can	have	1 point	of	intersection	$=1$
3	"	"	"	3 points	"	"	$=1+2$
4	"	"	"	6 "	"	"	$=1+2+3$
5	"	"	"	10 "	"	"	$=1+2+3+4$

Therefore to calculate the greatest possible number of points of intersection which a certain number of straight lines can have, add together the series of numbers from 1 up to but *not* including the number of lines.*

9. Find by calculation the greatest possible number of points of intersection among seven straight lines.
10. Calculate the same for eight straight lines.

* A still shorter method, shown in algebra, is to multiply the number of lines by 1 less than the number and divide the product by 2.

Thus, 10 lines will give $\frac{10 \times 9}{2}$ or 45 points of intersection.

11. If the greatest possible number of points of intersection among five straight lines is 10, what would the number be supposing that two of the lines were parallel? Draw a diagram.

3. To divide a group of points into two groups of various numbers.

1. In how many ways can two points be divided into two groups? Ans. One way: 1 — 1.
2. Three points? Ans. One way: 1 — 2.
3. Four points? Ans. Two ways: 1 — 3, 2 — 2.
4. Five points? Ans. Two ways: 1 — 4, 2 — 3.
5. Six points?
6. Seven points?
7. Eight points?
8. Nine points?

From the above results a rule can be formed: To find the number of ways in which a group of points can be divided into two groups, divide by 2 the number of points if it be an even number, or one less than the number of points if it be an odd number.

9. Calculate the number of ways in which 30 points can be divided into two groups.
10. Calculate the same for 35 points.
11. For 48 points.
12. For 27 points.

In these problems you will notice that you do not raise the question which of the two groups contains any particular point. Thus with three points *a*, *b*, and *c*, the groups *a* — *bc*, *b* — *ac*, and *c* — *ab*, all come under the head of one division.

4. To draw the greatest possible number of straight lines between points.

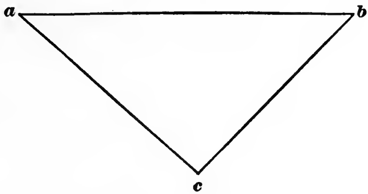
1. Between two points how many straight lines can be drawn?

Let *a* and *b* be the points.

Between *a* and *b* one straight line can be drawn, and only one. The straight line from *a* to *b* is the same as that from *b* to *a*.

2. Among three points how many straight lines can be drawn?

Let *a*, *b*, and *c* be the points.



Between *a* and *b* one straight line can be drawn; then the third point *c* can be connected with each of the other two points; therefore three straight lines can be drawn among three points.

3. According to the preceding problem, three is the *greatest* number of straight lines which can be drawn among three points: can you place three points so that not so many as three straight lines can be drawn among them?
4. What is the greatest number of straight lines which can be drawn among four points? Draw a diagram for three points and then proceed as in the second question.
5. Find the greatest number of straight lines for five points, drawing a diagram.
6. Do the same, with six points.

You will notice that in a group of points, for example, six, a straight line can be drawn from each of the six to each of the other five, thus making thirty lines; but the thirty must be divided by 2 to avoid counting each line twice, so that fifteen is the greatest possible number of different lines among six points.

This can be put in the form of a rule: To find the greatest possible number of different straight lines which can be drawn among a number of points, multiply the number of points by 1 less than that number, and divide by 2.

7. Find by calculation the greatest number of different straight lines which can be drawn among eight points.
8. Find the same for eleven points.
9. If three of the points in a group lie in one straight line, what change will there be in the total number of lines?
10. Can you arrange five points so that only one straight line can be drawn through them?

11. Can you arrange five points so that only five straight lines can be drawn among them?

12. Can you draw a diagram showing how you could plant seven trees so as to form six rows with three trees in each row?

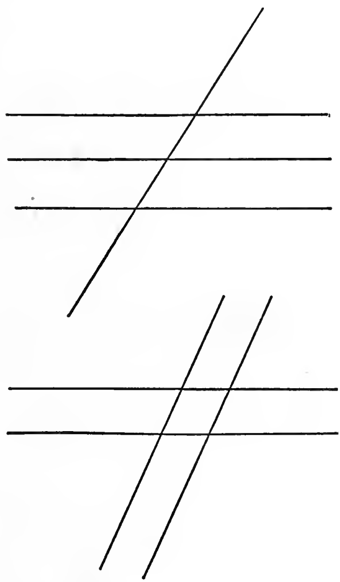
13. Can you draw a diagram showing how you could plant nineteen trees so as to form nine rows with five trees in each row? [Hint: draw two triangles so as to form a six-pointed star.]

14. Can you show how to plant nine trees in ten rows with three trees in each row? [Hint: begin by drawing a rectangle whose length is twice its width; then prolong the two shorter sides in opposite directions, each to a distance equal to its own length.]

CHAPTER XX

POINTS OF INTERSECTION

1. To find the number of points of intersection of straight lines which are divided in various ways into two groups, the lines of each group being parallel to one another.

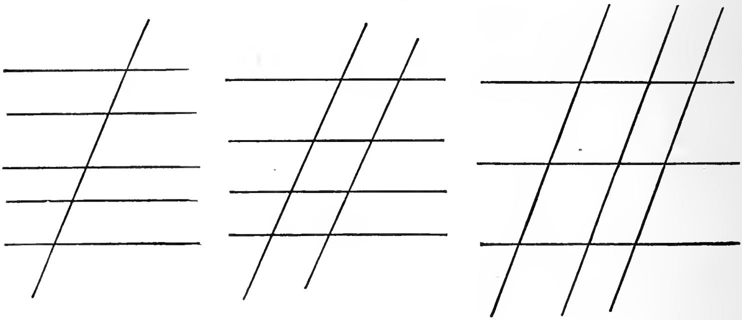


1. Suppose the number of lines to be four.

Four lines can be divided into two groups in two ways (see p. 134), 1 line and 3 lines, or 2 lines and 2 lines. If the lines of each group are parallel, how many points of intersection will there be?

You will notice that each line of a group cuts each line of the other group in one point; but the lines of the same group cannot cut one another: why?

2. How many points of intersection will six straight lines make when divided into groups as in the preceding problem?



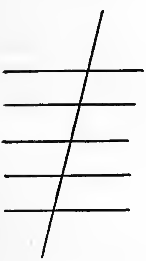
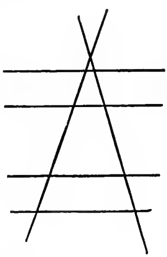
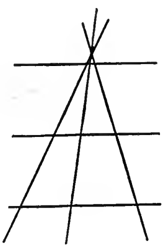
3. Five straight lines?
4. Eight straight lines?
5. Nine straight lines?
6. If the number of lines be 12, and the number of points of intersection be 27, how many lines are there in each group?
7. Can 11 lines and 17 lines each be divided into two groups of parallel lines so as to give 30 points of intersection?
8. What numbers of lines can be divided into groups, each of parallel lines, so as to give 30 points of intersection?
9. Fifteen lines, when divided in various ways into two groups of parallel lines, give the following numbers of points of intersection, — 14, 26, 36, 44, 50, 54, 56: what do you notice about the successive differences between these numbers?
10. The following is a table of the number of points of intersection of lines when divided in various ways into two groups of parallel lines: —

3	lines	make	2 points of intersection.			
4	"	"	3, or	4,		
5	"	"	4, or	6,		
6	"	"	5,	8, or 9,		
7	"	"	6,	10, or 12,		
8	"	"	7,	12,	15, or 16,	
9	"	"	8,	14,	18, or 20,	
10	"	"	9,	16,	21,	24, or 25.

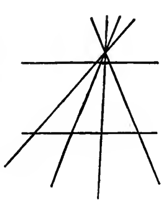
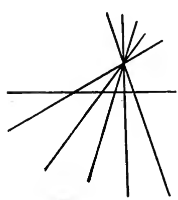
What do you notice about the increase in these numbers of points when you read the columns downwards?

11. Continue the table for 11 and 12 lines, using the above scheme as a guide.
12. How many lines parallel to a given straight line can be drawn through one point?
13. What is the greatest number of lines which can be drawn through four points, parallel to a given straight line?
14. Can you place four points so that only one line can be drawn through them parallel to a given straight line?
15. Can you place four points so that it would be impossible to draw a line through any two of them parallel to a given straight line?
16. How many straight lines parallel to each other can be drawn through one point?
17. Can you draw more than one set of parallel lines through two points?
18. What is the greatest and what is the least number of parallel lines which can be drawn through eight points?
19. Place three points so that one straight line can be drawn through them in the direction of northeast.
20. Place the three points so that two straight lines can be drawn through them in the direction of northeast.
21. Place the points so that no such line can be drawn through them.

2. To find the greatest number of points of intersection which can be made by a number of straight lines when divided into two groups in various ways, the lines of one group being parallel, and the lines of the other group cutting one another at one point.

1. Suppose the number of lines to be six. We have seen (p. 134) that six lines can be divided into two groups in three ways, 1 and 5, 2 and 4, 3 and 3. How many points of intersection can there be if one group consist of parallel lines, and the other group cut one another at one point?

In this problem either group in each case can consist of parallel lines or of lines cutting one another at one point. With six lines therefore there are five different arrangements. In each arrangement there will be a group of lines having no point of intersection among themselves since they are parallel, and there will be a group of lines having one point of intersection among themselves; and each line of one group will cut each line of the other group at one point. Therefore the total number of points of intersection will be found by adding 1 to the product of the numbers of lines in the two groups.

2. What is the greatest number of points of intersection among four straight lines divided into two groups, one group being parallel, and the other group cutting one another at one point?

3. Find the same for five straight lines.

4. Seven straight lines.

5. Eight straight lines.

6. If 12 lines be divided into groups of 8 and 4, how will the number of points of intersection when the larger group is parallel compare with the number when the smaller group is parallel?

7. Why is it that when 12 lines are divided into groups of 11 and 1, there will be one less point of intersection, or one more, according to the group which is made parallel?

8. If the point of intersection of the non-parallel group lie in the midst of the parallel group, will the total number of points of intersection be greater or less than if the point lie outside the parallels?

9. What if this point were to lie on one of the parallel lines?

10. What if a line of one group were parallel to a line of the other group?

11. Fifteen lines, when divided into two groups, the lines of one group being parallel, and the lines of the other group crossing one another at one point, make the following numbers of points of intersection, — 14, 15, 27, 37, 45, 51, 55, 57: what do you notice about the successive differences between these numbers?

12. The following is a table of the numbers of points of intersection made by lines divided into two groups, one group being parallel, and the other group crossing one another at one point: —

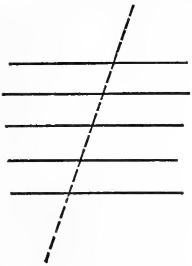
3	lines	make	2 or	3 points of intersection				
4	"	"	3,	4, or	5,			
5	"	"	4,	5, or	7,			
6	"	"	5,	6,	9, or 10,			
7	"	"	6,	7,	11, or 13,			
8	"	"	7,	8,	13,	16, or 17,		
9	"	"	8,	9,	15,	19, or 21,		
10	"	"	9,	10,	17,	22,	25, or 26.	

What do you notice about the increase in these numbers of points when you read each column downwards?

13. Continue the table for 11 and 12 lines, using the above scheme as a guide.

3. To find the greatest number of points of intersection which straight lines can have when divided into two groups in various ways, the lines of one group being parallel, and the lines of the other group cutting one another in the greatest number of points.

1. Suppose the number of lines to be six. These can be divided into groups of 5 and 1, 4 and 2, or 3 and 3. How many points of intersection can there be if one group consist of parallel lines, and the other group cut one another in the greatest number of points?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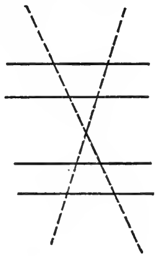
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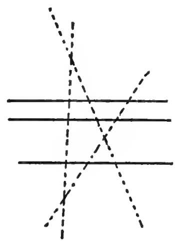
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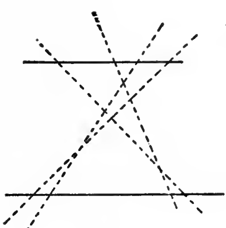
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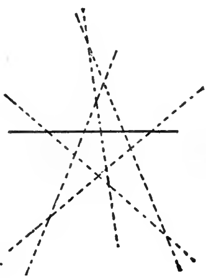
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In this problem either group in each case can consist of parallel lines or of lines cutting one another in the greatest number of points; with six lines, therefore, there are five arrangements. In each arrangement there will be a group of parallel lines, having no point of intersection, and a group of lines having the greatest number of such points which (see p. 133) can be found by multiplying the number of lines in the group by 1 less than that number and dividing the product by 2; also each line in one group can cut each line in the other group at one point.

Thus if the groups consist of two parallel lines and four lines cutting one another at the greatest number of points, the total number of points of intersection will be $\frac{4 \times 3}{2} + 8 = 14$ points.

The number of points for the five arrangements above will be 5, 9, 12, 14, and 15.

2. What is the greatest number of points of intersection which four straight lines can have when divided in various ways into two groups, the lines of one group being parallel, and those of the other group crossing each other at the greatest number of points?
3. Find the same for five straight lines.
4. Find the same for seven straight lines.
5. Calculate the numbers of points for twelve straight lines, without drawing diagrams.
6. If twenty lines were divided into groups of 14 and 6, would the total number of points of intersection be the same, whichever of the two groups were taken as parallel?
7. In the second diagram of the first question what change would there be in the answer if a line of one group were parallel to a line of the other group?
8. In the third diagram of the first question what change would there be in the answer if two lines of the non-parallel group were to cut one of the parallel lines at the same point?
9. If a straight line cut another straight line once, can the two meet again?
10. If fifteen straight lines be divided into two groups in various ways, the lines of one group being parallel and the lines of the other group cutting each other in the greatest number of points, the total numbers of points of intersection will be as follows,— 14, 27, 39, 50, 60, 69, 77, 84, 90, 95, 99, 102, 104, 105: what do you notice about the successive differences between these numbers?
11. The following is a table of the total numbers of points of intersection made by lines divided into groups as in the preceding question:—

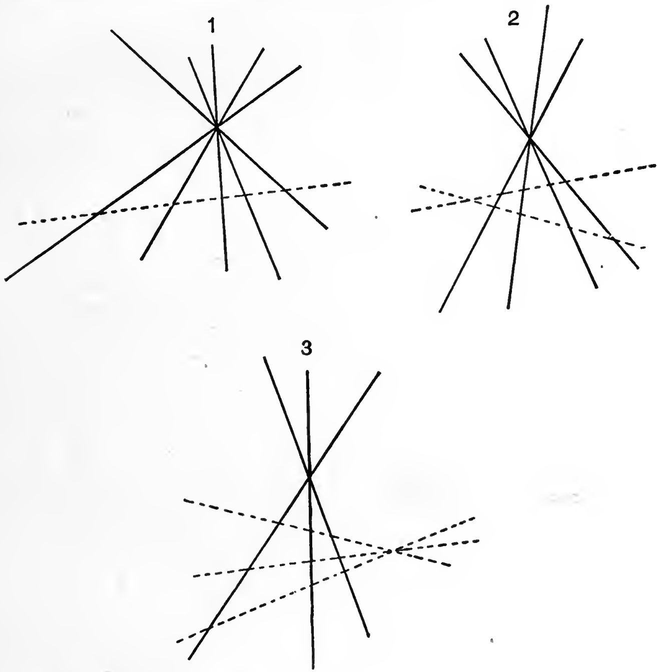
3 lines make	2 or	3 points of intersection.								
4 " "	3,	5, or	6,							
5 " "	4,	7,	9, or	10,						
6 " "	5,	9,	12,	14, or	15,					
7 " "	6,	11,	15,	18,	20, or	21,				
8 " "	7,	13,	18,	22,	25,	27, or	28,			
9 " "	8,	15,	21,	26,	30,	33,	35, or	36,		
10 " "	9,	17,	24,	30,	35,	39,	42,	44, or	45.	

What do you notice about the increase in these numbers of points when you read the columns downwards?

12. Continue the table for 11 and 12 lines, using the above scheme as a guide.

4. To find the greatest number of points of intersection which straight lines can have when divided in various ways into two groups, the lines of each group cutting one another at one point.

1. Suppose the number of lines to be six. These can be divided into groups of 5 and 1, 4 and 2, or 3 and 3. How many points of intersection can there be if the lines of each group cut one another in one point?



In this problem both groups in each case consist of lines cutting one another at one point; with six lines, therefore, there are three different arrangements. In each case there will be one point of intersection for the lines of each group among themselves; and each line of one group will cut each line of the other group. The total number of points of intersection, therefore, will be 2 more than the product of the numbers of lines in the two groups. Thus if the groups consist of two and four lines, the total number of points of intersection will be $2 + 2 \times 4 = 10$.

2. What is the greatest number of points of intersection which four straight lines can have when divided in various ways into two groups, the lines of each group cutting one another at one point?
3. Five straight lines?
4. Seven straight lines?
5. Eight straight lines?
6. In the second diagram of the first question how would the answer be affected if a line of one group were parallel to a line of the other group?
7. In the third diagram of the first question how would the answer be affected if the point of intersection of one group lay on a line of the other group?
8. On a map where towns are represented by single dots, and roads by single lines, there are two towns, from one of which three straight roads lead, and, from the other town, two straight roads. What is the greatest possible number of "cross-roads" among them?
9. In the eighth question what difference would there be if two of the roads were parallel?
10. In the eighth question what would the number be if one of the three roads from one town ran to the other town?
11. If 15 straight lines be divided into two groups in various ways, the lines of each group cutting one another at one point, the total numbers of points of intersection are as follows, — 15, 28, 38, 46, 52, 56, 58: what do you notice about the successive differences between these numbers?
12. The following is a table of the numbers of points of intersection made by lines when divided into groups as in the preceding question:—

3	lines	make	3 points of intersection.			
4	"	"	4, or	6,		
5	"	"	5, or	8,		
6	"	"	6,	10, or 11,		
7	"	"	7,	12, or 14,		
8	"	"	8,	14,	17, or 18,	
9	"	"	9,	16,	20, or 22,	
10	"	"	10,	18,	23,	26, or 27.

What do you notice about the increase in these numbers of points when you read the columns downwards?

13. Continue the table for 11 and 12 lines, using the above scheme as a guide.

CHAPTER XXI

ANGLES

I. ANGLES formed by two straight lines.

Draw two straight lines so that they may make: —

1. One angle.
2. Two angles.
3. Four angles.
4. Why cannot two straight lines form three angles?
5. Why cannot two straight lines form more than four angles?

Draw two straight lines so as to make: —

6. An acute angle.
7. A right angle.
8. An obtuse angle.
9. Can you increase the size of an angle by lengthening its sides?
10. If two straight lines extend from a point, one due east and the other northwest, what kind of an angle do they form?
11. Give the table of divisions of a right angle (see p. 39).

With the aid of a protractor draw two straight lines so as to make the following angles, and write against each angle its name — whether acute, right, or obtuse: —

12. 60°.	**16.** 55°.	**20.** 170°.
13. 100°.	**17.** 140°.	**21.** 10°.
14. 20°.	**18.** 85°.	**22.** 150°.
15. 90°.	**19.** 95°.	**23.** 30°.

24 How small can an acute angle be? How great?
25. How small can an obtuse angle be? How great?
26. Is there any variation in the size of a right angle?
27. If an acute angle be doubled, can the result be another acute angle? Can the result be a right angle? Can the result be an obtuse angle? Test your answers both by drawing diagrams and by giving the number of degrees in the angles.
28. If an obtuse angle be doubled, what will the result be? If a right angle? If an acute angle? Test your answers as in the preceding question.

29. Draw two straight lines so as to make an angle of 90°, and then prolong one of the lines through the vertex, thus forming another angle. How great is the second angle?

30. Draw two straight lines so as to make an angle of 60°, and then prolong one of the sides as before. With a protractor find the size of the second angle. What is the sum of the two angles?

31. Proceed in the same way, beginning with an angle of 105°.

32. Do the same, with an angle of 45°.

33. Do you find that, allowing for errors in measurement, the sum of the two angles is the same in each case? Is the sum 180°?

34. The *supplement* of an angle is the difference between that angle and two right angles: are the two angles in questions 29–32 each the supplement of the other?

35. Draw two straight lines so as to form at one point angles of 55° and 125°.

36. 150° and 30°.

37. 80° and 100°.

38. 95° and 85°.

39. If one of two angles formed by two straight lines be acute, what must the other be?

40. Can the following angles be formed at one point by two straight lines, — 110° and 85°? Draw a diagram to illustrate your answer.

41. If one of two angles formed at one point by two straight lines be 83° 20′, what is the other?

42. What is the supplement of 128° 40′ 20″?

43. What angle would be formed by the halves of the angles in the 30th question?

44. Would the answer to the preceding question be the same for the halves of any two angles whose sum is 180°?

45. The *complement* of an angle is the difference between that angle and a right angle: what is the complement of 20°? of 82°? of 17° 50′ 30″?

46. Draw two straight lines so as to make a right angle; then prolong each line through the vertex, thus forming three more angles: what is the size of these three angles? What is the sum in degrees of all four angles?

47. Draw two straight lines so as to make an angle of 60° and then prolong the sides as in the previous question; with a protractor find the size of each of the other angles. What is the sum of the four angles?

48. Proceed in the same way, beginning with an angle of 45°.

49. Do the same, with an angle of 105°.

50. Do you find that the sum of the four angles is the same in each case? Is it 360°?

51. Are the opposite angles in each case equal to each other?

52. Of how many different sizes are the four angles in any case?

53. Is there any one case where the four angles are of the same size?

54. Draw two straight lines so as to form two angles of 80° and two of 100°.

55. Draw two straight lines so as to form four angles as follows: 30°, 150°, 30°, 150°.

56. Draw two straight lines so as to form four angles, one of which is 20°.

Draw two straight lines so as to make:—

57. One right angle.
58. Two right angles.
59. Four right angles.
60. One acute angle.
61. One obtuse angle.
62. One acute and one obtuse angle.
63. Two acute and two obtuse angles.

2. Angles formed at one point by three straight lines.

Draw three straight lines so as to form at one point the following angles *:—

1. Two angles.
2. Three angles.
3. Four angles.
4. Five angles.
5. Six angles.

Draw three straight lines so as to form at one point the following groups of angles:—

6. 1 right and 1 acute.
7. 1 obtuse and 1 acute.
8. 2 acute.
9. 1 right and 2 acute.
10 1 obtuse and 2 acute.
11. 3 acute.
12. 1 right and 2 obtuse.
13. 1 acute and 2 obtuse.
14. 3 obtuse.
15. 2 obtuse and 2 acute.
16. 2 right, 1 obtuse, and 1 acute.
17. 3 right and 2 acute.
18. 2 obtuse and 3 acute.
19. 1 obtuse and 4 acute.
20. 1 obtuse, 1 right, and 3 acute.
21. 2 right and 4 acute.
22. 2 obtuse and 4 acute.
23. 6 acute.

3. Angles formed at two points by three straight lines.

Draw three straight lines so as to make at two points:—

1. Two angles.
2. Three angles.
3. Four angles.
4. Five angles.
5. Six angles.
6. Eight angles.
7. Why cannot three straight lines be drawn so as to make seven angles at two points?

Draw three straight lines so as to make at two points the following groups of angles:—

8. 1 right and 1 acute.
9. 1 right and 1 obtuse.
10. 1 acute and 1 obtuse.
11. 2 right.
12. 2 acute.
13. 2 obtuse.
14. 3 right.
15. 2 right and 1 acute.
16. 2 right and 1 obtuse.
17. 2 obtuse and 1 acute.

* It is understood that the angles are to be each less than 180°.

18. 2 acute and 1 obtuse.
19. 1 right, 1 acute, and 1 obtuse.
20. 4 right.
21. 2 obtuse and 2 acute.
22. 2 right, 1 acute, and 1 obtuse.
23. 5 right.
24. 4 right and 1 acute.
25. 4 right and 1 obtuse.
26. 1 right, 2 acute, and 2 obtuse.
27. 3 acute and 2 obtuse.
28. 3 obtuse and 2 acute.
29. 6 right.
30. 4 right, 1 acute, and 1 obtuse.
31. 2 right, 2 acute, and 2 obtuse.
32. 3 acute and 3 obtuse.
33. 8 right.
34. 4 right, 2 acute, and 2 obtuse.
35. 4 acute and 4 obtuse.

4. Angles formed at three points by three straight lines. Draw three straight lines so as to make at three points:

1. Three angles.
2. Four angles.
3. Five angles.
4. Six angles
5. Seven angles.
6. Eight angles.
7. Nine angles.
8. Ten angles.
9. Twelve angles.
10. Why cannot eleven angles be formed in this way?

Draw three straight lines so as to form at three points the following groups of angles: —

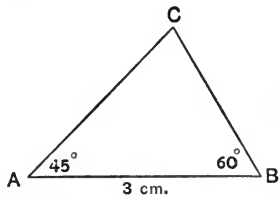
11. 3 acute.
12. 1 right and 2 acute.
13. 2 acute and 1 obtuse.
14. 2 right and 2 acute.
15. 1 right, 2 acute, and 1 obtuse.
16. 3 acute and 1 obtuse.
17. 2 acute and 2 obtuse.
18. 2 right, 2 acute, and 1 obtuse.
19. 1 right, 2 acute, and 2 obtuse.
20. 3 acute and 2 obtuse.
21. 3 obtuse and 2 acute.
22. 4 right and 2 acute.
23. 2 right, 2 acute, and 2 obtuse.
24. 1 right, 3 acute, and 2 obtuse.
25. 4 acute and 2 obtuse.
26. 3 acute and 3 obtuse.
27. 4 right, 2 acute, and 1 obtuse.
28. 2 right, 3 acute, and 2 obtuse.
29. 1 right, 3 acute, and 3 obtuse.
30. 4 acute and 3 obtuse.
31. 3 acute and 4 obtuse.
32. 4 right, 2 acute, and 2 obtuse.
33. 2 right, 3 acute, and 3 obtuse.
34. 4 acute and 4 obtuse.
35 4 right, 3 acute, and 2 obtuse.
36. 1 right, 4 acute, and 4 obtuse.
37. 5 acute and 4 obtuse.
38. 4 acute and 5 obtuse.
39. 4 right, 3 acute, and 3 obtuse.
40. 2 right, 4 acute, and 4 obtuse.
41. 5 acute and 5 obtuse.
42. 4 right, 4 acute, and 4 obtuse.
43. 6 acute and 6 obtuse.

CHAPTER XXII

TRIANGLES, QUADRILATERALS, AND POLYGONS

1. REVIEW what is said about triangles on pp. 34–36.

1. Construct a triangle having one side 3 cm. long, and the angles at the end of that side 60° and 45°.



Draw a straight line AB 3 cm. long. At A draw a line so as to make with AB an angle of 45°; and at B draw a line so as to make with AB an angle of 60°; prolong these lines until they meet at C; then ABC will be the triangle.

Measure with the protractor the angle C.

What is the sum of the angles A, B, and C?

2. Construct a triangle having one side 5 cm. long, and the angles at the end of that side 30° and 50°.

Measure the third angle, and find the sum of all three angles.

3. Do the same, taking for the side 4 cm., and for the angles, 120° and 40°.

4. Do the same, taking for the side 4 cm., and for the angles, 20° and 40°.

5. Take the side 5 cm., and the angles 70° and 20°.

Allowing for inaccuracy in measuring the angles, do you find that these five triangles agree in the sum of their angles? Is the sum 180°?

6. Construct a triangle having one side 4 cm. and the angles at the ends each 40°. Measure the third angle, find the sum of all three angles, and compare the lengths of the sides opposite the equal angles.

7. Do the same, taking for the side 5 cm., and for the equal angles 30°.

8. Do the same, taking for the side 5 cm., and for the equal angles 45°. From the last three triangles, what do you find about the equality of sides when there are two equal angles in a triangle?

9. Construct a triangle having a side 5 cm. long, and the angles at the end each 60°. What do you find about the three angles and the three sides of this triangle?
10. Construct a triangle having a side 8 cm. long, and the angles at the ends, 30° and 60°. Measure the third angle and the other two sides.
    (*a*) Does the longest side lie opposite the greatest angle?
    (*b*) Does the shortest side lie opposite the smallest angle?
    (*c*) 60° is twice 30°; but is the side opposite 60° twice as long as the side opposite 30°?
    (*d*) Is there any side twice as long as the side opposite 30°?
11. What is the sum of the angles of any triangle?
12. If the three angles are equal, how many degrees are there in each?
13. How many of the angles of a triangle can be obtuse?
14. How many of the angles can be right angles?
15. Construct a triangle having three acute angles.
16. Construct a triangle having one obtuse and two acute angles.
17. Construct a triangle having one right and two acute angles.

QUADRILATERALS

2. Review what is said about quadrilaterals on pp. 20–22. Construct quadrilaterals whose angles shall be as follows:

1. 4 right.
2. 2 right, 1 acute, and 1 obtuse.
3. 1 right, 2 acute, and 1 obtuse.
4. 1 right, 1 acute, and 2 obtuse.
5. 3 acute and 1 obtuse.
6. 2 acute and 2 obtuse.
7. 1 acute and 3 obtuse.
8. 90°, 90°, 90°, 90°. Let all the sides be equal. What is the name of this figure?
9. 90°, 90°, 90°, 90°. Make a figure which shall not have all its sides equal. Notice whether any of the sides are equal or parallel. What is the name of this figure?
10. 90°, 90°, 160°, 20°. Let the figure have two parallel sides. What is its name?
11. 90°, 90°, 160°, 20°. Let the figure have no parallel sides. What is its name?
12. 100°, 80°, 100°, 80°. Arrange the angles so that the figure may be a parallelogram.
13. Arrange the angles of the preceding problem so that the figure may be a trapezoid.
14. 150°, 30°, 150°, 30°. Let the figure be a parallelogram.
15. Change the figure of the previous problem into a rhombus.
16. What is the difference between a rhombus and a parallelogram?

17. Could the sides of a rhombus and those of a parallelogram be equal, each to each?
18. What is the difference between a rectangle and a parallelogram?
19. Could the sides of a rectangle be equal, each to each, to those of a square?
20. What is the difference between a square and a rectangle?
21. What is the difference between a rhombus and a square?
22. Could the sides of a rhombus and those of a square be equal, each to each?
23. In what particular respect do the square and rectangle agree?
24. In what do the rhombus and square agree?
25. What can be said alike of all four figures, — rhombus, square, rectangle, and parallelogram?

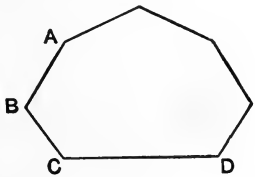
POLYGONS

3. Review what is said about polygons on pp. 72–78.

How many sides have the following polygons? —

1. Quadrilateral.
2. Pentagon.
3. Hexagon.
4. Heptagon.
5. Octagon.
6. Nonagon.
7. Decagon.
8. Dodecagon.
9. Pentedecagon.
10. Icosagon.

The *angles* of a polygon are the angles made by its sides meeting, as *ABC*, *BCD*, etc.



They are measured within the polygon, and are sometimes called interior angles.

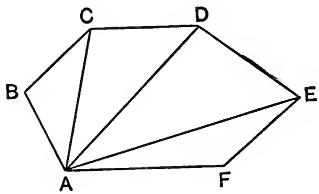
How many angles have the following polygons: —

11. Quadrilateral.
12. Pentagon.
13. Hexagon.
14. Heptagon.
15. Octagon.
16. Nonagon.
17. How does the number of angles of a polygon compare with the number of sides?
18. How many angles has a dodecagon?
19. An icosagon?
20. A polygon of thirty sides?

The *vertices* of a polygon are the vertices of its angles, as *A*, *B*, *C*, etc.

How many vertices have the following polygons?

21. Rhombus.
22. Pentagon.
23. Hexagon.
24. Heptagon.
25. Octagon.
26. Nonagon.
27. Decagon.
28. Dodecagon.
29. How does the number of vertices compare with the number of angles of a polygon? With the number of sides?
30. How many vertices has a polygon of forty sides?



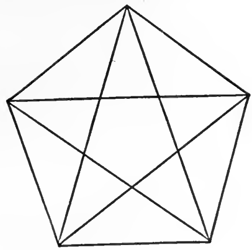
A *diagonal* of a polygon is a straight line which joins any two vertices, except those already connected by the sides, as *AC*, *AD*, etc. If all possible diagonals be drawn from any one vertex, as *A*, the polygon will be divided into triangles, *ABC*, *ACD*, etc.

Into how many triangles can the following polygons be divided by drawing diagonals from any vertex: —

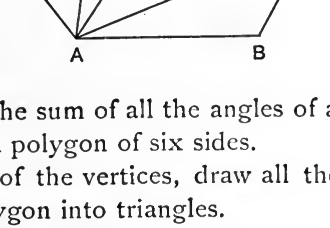
31. Quadrilateral.
32. Pentagon.
33. Hexagon.
34. Heptagon.
35. Octagon.
36. Nonagon.
37. Decagon.
38. Triangle.
39. The number of triangles is always less than the number of sides by the same amount: how much less? why?
40. Into how many triangles can a polygon of thirty sides be divided by drawing diagonals from any vertex?

Draw all possible diagonals in the following polygons, and find the number in each case.

41. Quadrilateral.
42. Hexagon.
43. Heptagon.
44. Octagon.



45. The number of diagonals which can be drawn from any one vertex is always less than the number of sides by the same amount: how much less? why?
46. If you were to multiply the number of diagonals from one vertex by the number of vertices, the product would be greater than the number of *different* diagonals: how many times greater?
47. What rule can you give for finding the total number of different diagonals in any polygon?
48. Calculate the total number of diagonals in a polygon of twenty sides.
49. Calculate the same for a polygon of thirty sides.
50. Calculate it for a polygon of forty-eight sides.



4. To find the sum of all the angles of a polygon.

ABCDEF is a polygon of six sides.

From *A*, one of the vertices, draw all the diagonals, thus dividing the polygon into triangles.

1. How does the number of triangles compare with the number of sides?
2. Can you see that the sum of the angles of the triangles is the same as the sum of the angles of the polygon?
3. What is the sum of the angles of any triangle?
4. What is the sum of the angles of all the triangles, *ABC*, *ACD*, etc., together?
5. What, then, is the sum of all the angles of the polygon *ABCDEF*?
6. If the sum is eight right angles, what is the sum in degrees?

7. Complete the following table, showing the number of triangles which compose the polygons, and the sum of their angles: —

3 sides,	1 triangle,	2 right angles.	
4 "	2 triangles,	4 "	
5 "	"	"	
6 "	"	"	
7 "	"	"	

What do you notice about the increase of the numbers when you read the columns downwards?

From the preceding results a rule can be derived: —

To find the sum of the angles of any polygon, subtract 2 from the number of sides and double the remainder: the result will be the sum of the angles expressed in right angles; the result multiplied by 90 will be the sum of the angles expressed in degrees.

Calculate the sum of the angles of the following polygons, expressing the results both in right angles and in degrees:

8. Octagon.
9. Nonagon.
10. Decagon.
11. Dodecagon.
12. Pentedecagon.
13. Icosagon.
14. Eighteen sides.
15. Twenty-four sides.
16. Twenty-five sides.
17. Thirty sides.
18. Thirty-two sides.
19. Forty sides.
20. Forty-eight sides.

Since the angles of a regular polygon are equal, the size of one of the angles can be found by dividing the sum of all the angles by the number of sides of the polygon.

Find in degrees the size of one angle of the following regular polygons: —

21. Pentagon.
22. Hexagon.
23. Heptagon.
24. Octagon.
25. Nonagon.
26. Decagon.
27. Dodecagon.
28. Pentedecagon.
29. Icosagon.
30. Thirty-two sides.

5. **Pentagons and Hexagons.** Pentagons are possible with ten different combinations of obtuse, right, and acute angles.

Particular care should be taken in their construction to produce figures which shall be distinct in the sizes of the angles as well as neat.

Construct pentagons which shall have the following angles: —

31. 5 obtuse.
32. 4 obtuse, 1 right.
33. 4 obtuse, 1 acute.
34. 3 obtuse, 2 right.
35. 3 obtuse, 1 right, 1 acute.
36. 3 obtuse, 2 acute.
37. 2 obtuse, 3 right.
38. 2 obtuse, 2 right, 1 acute.
39. 2 obtuse, 1 right, 2 acute.
40. 2 obtuse, 3 acute.

Hexagons are possible with ten different combinations of obtuse, right, and acute angles.

Here, also, particular care should be taken to produce distinct, neat, symmetrical figures.

Construct hexagons which shall have the following angles: —

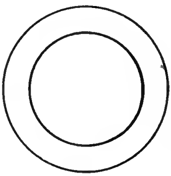
41. 6 obtuse.
42. 5 obtuse, 1 right.
43. 5 obtuse, 1 acute.
44. 4 obtuse, 2 right.
45. 4 obtuse, 2 acute.
46. 4 obtuse, 1 right, 1 acute.
47. 3 obtuse, 2 right, 1 acute.
48. 3 obtuse, 1 right, 2 acute.
49. 3 obtuse, 3 right.
50. 3 obtuse, 3 acute.

CHAPTER XXIII

CIRCLES

1. **Positions of Circles with Reference to one another.** Review what is said about circles on pp. 87–96.

1. Two circles may have the same centre, in which case they are called *concentric* circles and their circumferences have no points in common.



2. Two circles may have different centres. Draw diagrams to illustrate the following cases: —

(*a*) One circle lying wholly within the other with the circumferences having no point in common.
(*b*) One circle lying wholly within the other with the circumferences having one point in common.
(*c*) One circle lying partly within the other, the circumferences having two points in common.
(*d*) The circles lying wholly outside one another with the circumferences having one point in common.
(*e*) The circles lying wholly outside one another with the circumferences having no point in common.

1. Draw two concentric circles so that the radius of one may be equal to a diameter of the other.
2. Draw two circles so that the centre of each may lie on the circumference of the other.

Draw two circles with their centres at the ends of a straight line AB, so that —

3. Their areas may have no point in common.
4. Their areas may have one point in common.
5. The area of one may be included in the area of the other.

6. Draw a circle; then draw two more circles inside the first, each having a diameter equal to a radius of the first circle.
7. Draw three concentric circles so that the radius of the greatest may be equal to the sum of the radii of the other two.

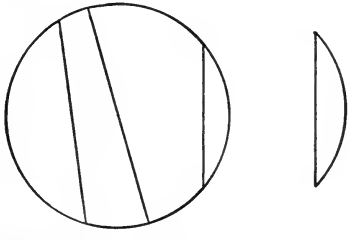
Draw two circles with their centres at the ends of a straight line AB, so that their circumferences may have —

8. No point in common.
9. One point in common.
10. Two points in common.

Draw two circles so that the distance between their centres may be —

11. Equal to the sum of their radii.
12. Less than the sum of their radii.
13. o.
14. Greater than the difference between their radii.
15. Equal to the difference between their radii.
16. Less than the difference between their radii.

17. Draw three circles of equal radii, with their centres in a straight line, so that the circumference of the middle circle may pass through the centres of the other two.
18. Draw three unequal circles, two inside the third, with their centres in a straight line, so that the radius of one may be equal to the sum of the radii of the other two.
19. Draw three equal circles with their centres in a straight line which is equal to the sum of their diameters.
20. Draw three circles so that the centres of two may each lie in two of the other circumferences.



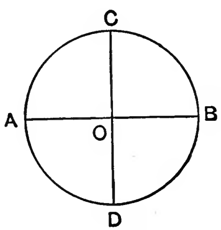
2. **Chords of Circles.** A *chord* is a straight line which connects the ends of an arc. The chord is said to *subtend* the arc. The word originally meant the string of a musical instru-

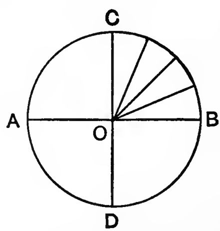
ment such as a harp; it is the same as cord, but has a different spelling in geometry.

1. Draw a chord which shall be equal to a radius of the circle.
2. Draw a chord through the centre of the circle. What special name do you give to this chord?
3. Draw the longest chord you can in a circle. What do you notice about this chord?
4. Draw two unequal chords perpendicular to each other.
5. Draw a chord of any length. Draw diameters through its ends, and three other chords through the ends of the diameters. What shape has the quadrilateral formed by the four chords?
6. If AB is the diameter of a circle, where is the centre?
7. Draw a diameter. Then draw four chords of various lengths, each perpendicular to this diameter? What do you notice about the parts into which the diameter divides each chord?
8. If you were to draw a perpendicular at the middle of a chord, through what particular point of the circle would it pass?
9. The last two questions suggest methods of finding the centre of a circle when the centre is not indicated in the figure.
Do you see how it could be done?
10. Through one point in a circumference how many chords of the same length can you draw?
How many diameters?

3. **Subdivision of a Circumference into Arcs.** AB and CD are diameters drawn perpendicular to each other. You will notice that they divide the circumference into four equal arcs.

Also, if radii be drawn dividing the right angle BOC into four equal angles, the arc BC will be divided into four equal arcs, each arc corresponding to one of the four angles.

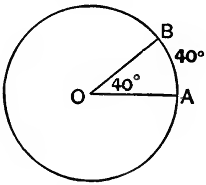




Just as the right angle BOC can be divided into ninety equal parts, each of $1°$, so can the arc BC be divided into

ninety equal parts, each called an arc of 1°; and the arc of 1° is subdivided into arcs of 1′ and 1″. The whole circumference, therefore, consists of 360 parts, each of which is an arc of 1°.

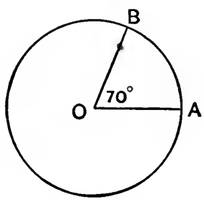
This idea is expressed by saying that "an angle at the centre is measured by the arc between its sides;" which



means that an angle formed by two radii is the same part of four right angles as the arc between the ends of the radii is of the whole circumference. Thus, if the angle AOB is 40°, the arc AB is also 40°.

To mark off an arc of required size upon the circumference of a given circle.

1st. With the aid of a protractor.



Let O be the centre of the given circle and 70° be the required arc.

Draw OA and OB, radii forming an angle of 70°.

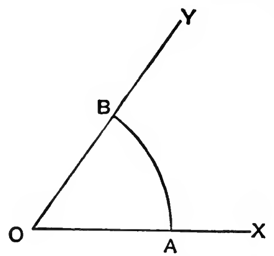
Then AB will be the required arc.

Draw circles with any convenient radii, and mark off the following arcs, one on each circumference, with the aid of a protractor: —

1. 20°. **2.** 50°. **3.** 80°. **4.** 140°. **5.** 160°.

An arc of any required size can be constructed without drawing the whole circumference.

If the required arc be 55°, construct an angle *XOY* of 55°. Then with the vertex *O* as a centre, and a radius equal to the radius of the circle, draw the arc *AB* between the sides of the angle. This will be the required arc.



Construct the following arcs without drawing more of the circumference in any case: —

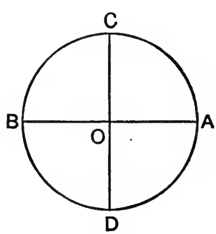
6. 40°. **7.** 65°. **8.** 100°. **9.** 115°. **10.** 130°. **11.** 120°.

2d. With the aid of compasses.

Certain arcs can be constructed more rapidly and neatly with the aid of compasses than with the aid of a protractor.

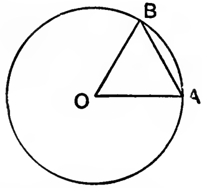
Some of the principal cases are the following: —

(*a*) To construct an arc of 90°.



Draw two diameters, *AB* and *CD*, perpendicular to each other. Then any one of the four arcs thus marked off will be the required arc of 90°; for each is one-fourth of the whole circumference.

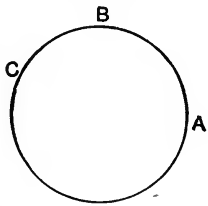
(*b*) To construct an arc of 60°.



Draw the chord AB equal to the radius of the circle. Then the arc AB is the required arc of 60°.

For if you draw the radii OA and OB, the triangle AOB will be equilateral, each side being equal to a radius; each angle, therefore, is equal to 60°; and if the angle O is 60°, its corresponding arc AB is also 60°.

(*c*) To construct an arc of 150°.

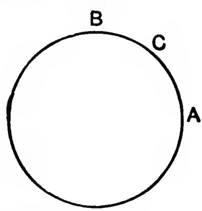


First mark off an arc AB equal to 90°.

Then, beginning at B mark off an arc BC equal to 60°. The arc AC will be the required arc of 150°.

For $AC = AB + BC = 90° + 60° = 150°$.

(*d*) To construct an arc of 30°.

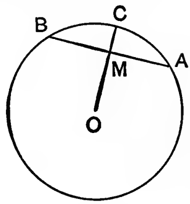


First mark off an arc AB equal to 90°.

Then mark off a part of AB, namely, AC, equal to 60°. CB will be the required arc of 30°.

For $CB = AB - AC = 90° - 60° = 30°$.

(*e*) To construct an arc of 45°.



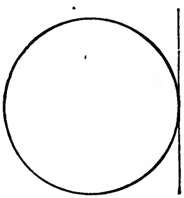
First mark off an arc AB equal to 90°, and draw its chord. Then draw the radius OC through the middle point M of the chord AB. The arcs AC and CB will each be equal to the required arc of 45°. For a radius (or diameter) which passes through the middle of a chord will also pass through the middle of the arc subtended by the chord.

Thus $AC = CB =$ one-half of $90° = 45°$.

With the aid of compasses construct the following arcs:

11. 15°. **12.** 75°. **13.** 105°. **14.** 120°. **15.** 135°.
16. 7° 30′. **17.** 37° 30′. **18.** 52° 30′. **19.** 97° 30′. **20.** 67° 30′.

4. **Tangents.** A *tangent* is a straight line which has only one point in common with a circumference, however far the line may be prolonged.



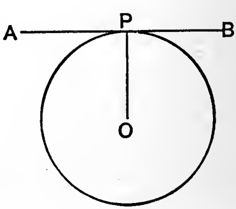
The word tangent means "touching."

Besides the definition there are two facts to be noticed about a tangent.

1st. A tangent has the same direction as the circumference at the point of contact.

The curves of railroads depend on this fact, as was explained on p. 92; the straight rails are tangent to the curved ones at the points where they meet.

2d. A tangent is perpendicular to the radius (or diameter) drawn to the point of contact.

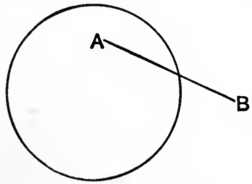


This affords a method of drawing a tangent, when you know the point of tangency.

Suppose that you wish to draw a tangent at the point P. First draw the radius OP. Then,

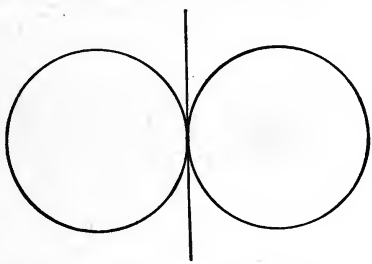
at P draw the straight line AB perpendicular to OP. AB will be the required tangent.

1. Draw a circle; take any point in the circumference and construct a tangent at that point.
2. In the annexed figure, AB is not a tangent to the circle: why not?



3. Draw a tangent at each end of some diameter, and compare their directions.
4. Draw two diameters perpendicular to each other, and then draw a tangent at each end of the diameters; prolong the tangents until they meet. What shape has the figure formed by the tangents?
5. Find three points on a circumference, so situated that the three arcs into which the circumference is divided may each be 120°. Then draw a tangent at each point, and prolong the three until they meet. What shape has the figure formed by the tangents?

Two circles are said to be tangent to each other when they can be tangent to the same straight line at the same point.

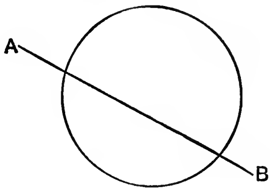


6. In the preceding diagram the circles are called tangent *externally*, because one circle lies outside the other. How does the distance between their centres compare with the two radii?
7. Draw a diagram showing two circles tangent *internally*, — that is, one circle lying within the other. How does the distance between their centres compare with the two radii?
8. Draw a diagram showing three circles all tangent at the same point. Are the three centres and the point of contact all in the same straight line?
9. How would you draw a line through a point which lies in a given circumference, so that it would contain the centres of all circles which could be tangent to the given circle at that point?

10. If a circle has a radius of 14 mm., how would you draw a line which would contain the centres of all circles which have a radius of 8 mm. and are tangent externally to the first circle?

5. **Secants.** A *secant* is a straight line which cuts a circumference in two points, as *AB*.

The word secant means "cutting."



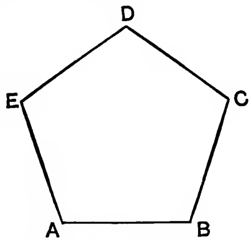
If a line passes beyond the circumference at only one point, it is still considered to be a secant. In fact, the *length* of a secant in many problems is understood to be the distance from the point outside the circle, where it begins, up to the second point where it reaches the circumference.

1. If you prolong a chord, what does it become?
2. Draw a chord of a circle and a secant whose length is equal to that of the chord.
3. Why cannot a tangent be changed into a secant by prolonging it?
4. From a point outside a circle draw four secants each ending where it reaches the circumference the second time. Are those secants longer or shorter which pass near the centre of the circle?
5. How would you draw the longest possible secant from a point outside a circle?
6. From a point outside a circle how would you draw a secant ending at the second point where it meets the circumference, so that the least possible part may lie outside the circle?
7. In two concentric circles draw a line which shall be a chord of one and a secant of the other.
8. In two concentric circles draw a line which shall be a secant of one and the longest possible chord of the other.
9. In two intersecting circles draw a line which shall be a chord of both. Then change this common chord into a common secant having twice the length of the chord.
10. Draw two circles which shall be tangent externally. Then draw a line with both ends in the circumferences, so that it may be a secant of each circle and equal to the sum of their diameters.

CHAPTER XXIV

REGULAR POLYGONS

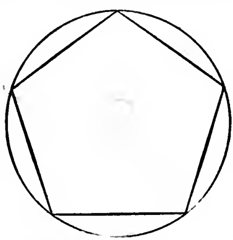
1. A *regular polygon* is a polygon which is both equilateral and equiangular. See p. 73.



A Regular Polygon

The construction of regular polygons is most easily effected with the aid of circles and depends on the following truths:

1st. If a circumference be divided into equal arcs, the chords of those arcs will be equal, and the angles formed by

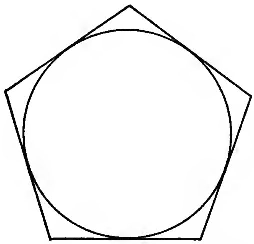


Inscribed Regular Polygon

the chords will also be equal; the polygon, therefore, thus formed will be regular. The polygon is then said to be *inscribed in* the circle.

Any polygon, whether regular or not, is called an inscribed polygon when its sides are all chords of a circle.

2d. If a circumference be divided into equal arcs, the tangents drawn at the points of division of the arcs, and prolonged until they meet one another, will be equal, and the



Circumscribed Regular Polygon

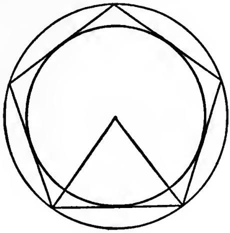
angles formed by the tangents will also be equal; the polygon, therefore, thus formed will be regular. The polygon is then said to be *circumscribed about* the circle.

Any polygon, whether regular or not, is called a circumscribed polygon when its sides are all tangents to a circle.

To construct a regular polygon, therefore, of any number of sides; first draw a circle and divide the circumference into as many equal parts as the polygon is to have sides, with the aid of compasses or protractor as shown on p. 159; then at the points of division of the arcs draw chords or tangents, according as the polygon is to be inscribed or circumscribed.

Construct regular inscribed and circumscribed polygons of the following numbers of sides, using one circle for two polygons of the same number of sides:—

1. Triangle, inscribed.
2. Triangle, circumscribed.
3. Square, inscribed.
4. Square, circumscribed.
5. Pentagon, inscribed.
6. Pentagon, circumscribed.
7. Hexagon, inscribed.
8. Hexagon, circumscribed.
9. Octagon, inscribed.
10. Octagon, circumscribed.



The *centre* of a regular polygon is the same point as the centre of its inscribed and circumscribed circles.

An *angle at the centre* of a regular polygon is the angle formed by two lines drawn from the centre of the polygon to two consecutive vertices. In any regular polygon this angle is equal to 360° divided by the number of sides of the polygon.

Find in degrees the size of an angle at the centre of the following regular polygons: —

1. Triangle. **2.** Square. **3.** Pentagon. **4.** Hexagon. **5.** Heptagon. **6.** Octagon. **7.** Nonagon. **8.** Decagon. **9.** Pentedecagon. **10.** Icosagon.

2. To find the length of the circumference of a circle.

The length of a curved line is usually difficult to find by actual measurement. Sometimes you can apply a flexible ruler, such as a tape, which will bend so as to follow the curve. Usually, however, the length of a curve is found by calculations which depend on the nature of the particular curve in question. For that reason engineers and makers of machinery are careful to use curves whose nature is known.

The circumference of a circle is one of the curves whose length is most easily calculated. Geometers have proved that the length of a circumference is a little more than three times the length of its diameter; that is, if a diameter is 2 inches, the circumference will be a little more than 6 inches long.

You can test this by bending a strip of paper around the curved surface of a cylinder, noting the length, and comparing it with the length of the diameter of the base.

Make the following calculations, supposing the length of a circumference to be three times the length of its diameter:

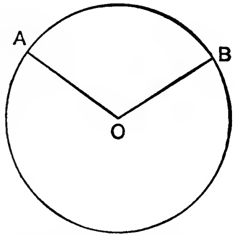
1. Diameter = 2 cm.; circumference = ?
2. " = 3 " " = ?
3. " = 4 " " = ?
4. " = 2 inches; " = ?
5. Radius = 1 cm.; " = ?
6. " = 2 " " = ?
7. Circumference = 6 cm.; diameter = ?; radius = ?
8. " = 9 " " = ? " = ?
9. " = 3 inches; " = ? " = ?
10. " = 12 " " = ? " = ?

Geometers have proved that the *exact* comparison between a circumference and its diameter cannot be expressed in numbers; and they have accordingly agreed to denote it by the Greek letter π (pronounced *pie*). This is expressed by saying that a circumference has π times the length of its diameter. π is *nearly* equal to $3\frac{1}{7}$; that is, if a diameter is 5 cm. long, the circumference will be 5π or about $15\frac{5}{7}$ cm. long.

Make the following calculations, considering π to be equal to $3\frac{1}{7}$: —

11. Diameter = 1 cm.; circumference = ?
12. " = 2 " " = ?
13. " = 3 " " = ?
14. " = 7 " " = ?
15. Radius = 1 inch; " = ?
16. " = 2 inches; " = ?
17. " = 3 " " = ?
18. Circumference = 22 cm.; diameter = ?
19. " = 44 " radius = ?
20. " = 11 inches; " = ?

3. **To find the Length of an Arc.** To calculate the length of an arc, you must know the size of the arc in degrees and the length of the circumference of which the arc is a part.



Suppose the arc AB to be $70°$, and the diameter of the circle to be 3 cm.

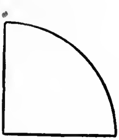
1st. The whole circumference is $3\,\pi$, or $3 \times 3\frac{1}{7}$, or $9\frac{3}{7}$ cm. long.

2d. As the arc is $70°$ and the whole circumference contains $360°$, the arc is $\frac{70}{360}$ or $\frac{7}{36}$ of the circumference.

$\therefore$ The length of the arc is $\frac{7}{36} \times 9\frac{3}{7}$, or $\frac{7}{36} \times \frac{66}{7}$, or $\frac{11}{6}$, or $1\frac{5}{6}$ cm. long.

Calculate the lengths of the following arcs, considering π to be equal to $3\frac{1}{7}$.

1. Arc 35°, diameter of circle 1 cm.
2. Arc 60°, " " 7 cm.
3. Arc 70°, " " 14 cm.
4. Arc 140°, radius of circle 35 mm.
5. Arc 90°, " " 4 cm.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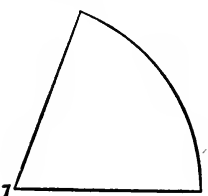
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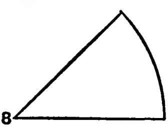
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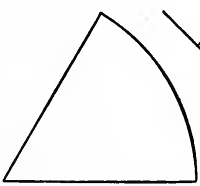
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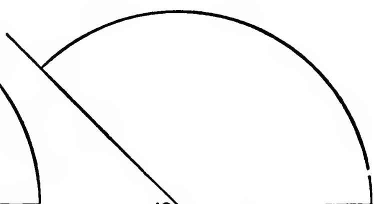
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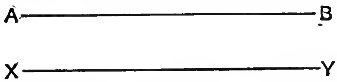
CHAPTER XXV

CONSTRUCTIONS

1. To construct a straight line which shall be equal to a given straight line.

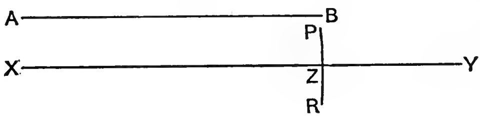
(*a*) With the aid of a graduated ruler: —



Let AB be the given straight line:

Measure the length of AB with a graduated ruler, and then draw XY of the same length. XY will be the required line.

(*b*) With the aid of compasses and an ungraduated ruler: —



Let AB be the given straight line.

Draw XY a straight line of any convenient length evidently greater than AB.

With X as a centre, and a radius equal to AB, draw an arc PR crossing XY at Z.

XZ will be the required straight line.

Construct straight lines equal to the following, with the aid of compasses and ungraduated ruler: —

1 6

2 7

3 8

4 9

5 10

2. **To bisect a Given Straight Line.** To *bisect* means to divide into two equal parts.

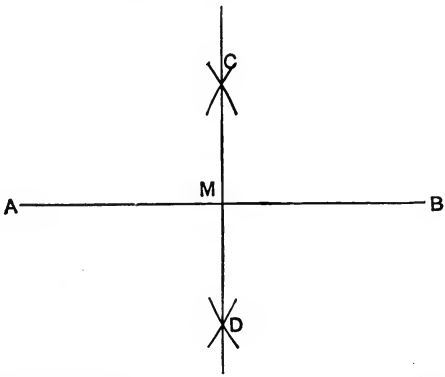
(*a*) With the aid of a measuring ruler.

Let AB be the given line.

Applying a graduated ruler to AB, you will find the length to be 6 cm.

Divide this length by 2, and measure off the quotient 3 cm. from A or from B to the point M, which will be the middle point of AB.

(*b*) With the aid of compasses and ungraduated ruler.

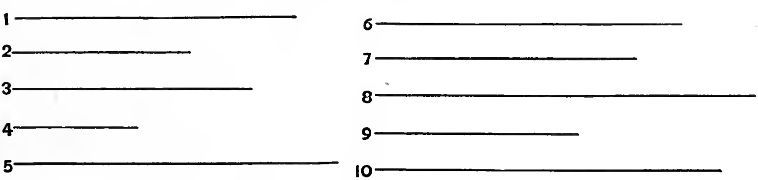


Let AB be the given line.

With A and B as centres, and any convenient radius which is evidently greater than one-half AB, draw arcs crossing one another at C and D on opposite sides of AB.

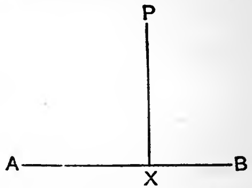
Connect C and D with a straight line crossing AB at M, which will be the middle point of AB.

Construct straight lines equal to the following, and bisect them with the aid of compasses and ungraduated ruler.



3. To construct a perpendicular from a given point to a given straight line.

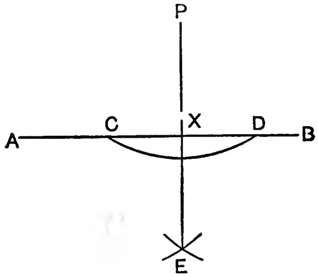
(*a*) With the aid of a square:—



Let P be the given point, and AB the given straight line.

Apply a square so that one edge of the right angle may be close to AB, and the other edge close to P; along the latter edge draw a line PX to AB. PX will be the required perpendicular.

(*b*) With the aid of compasses and ungraduated ruler.



Let P be the given point, and AB the given straight line.

With P as a centre, and any convenient radius evidently longer than the perpendicular distance from P to AB, draw an arc cutting AB at C and D.

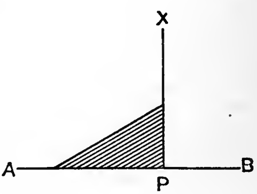
With C and D as centres and any convenient radius evidently longer than one-half CD, draw arcs crossing one another at E.

Draw the straight line PE crossing AB at X.

PX will be the required perpendicular.

4. To construct a perpendicular to a given straight line from a given point in the line.

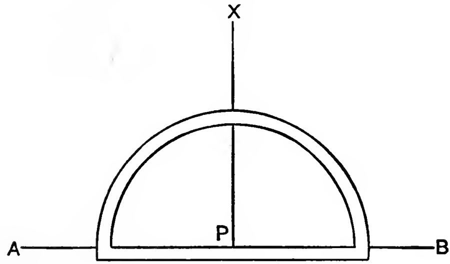
(*a*) With the aid of a square:—



Let AB be the given straight line and P the given point in AB.

Apply a square so that the vertex of the right angle may be at P, and one edge may lie close to AB. Along the other edge draw PX, which will be the required perpendicular.

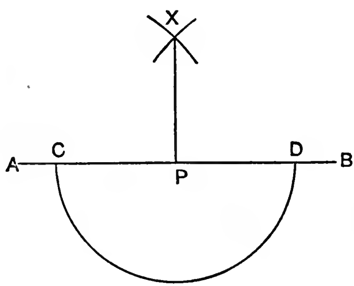
(*b*) With the aid of a protractor and ruler.



Let AB be the given straight line, and P the given point in AB.

Apply the straight edge of the protractor to AB so that the notch may be at P. Then draw PX so as to make the angle BPX equal to 90°. XP will be the required perpendicular.

(*c*) With the aid of compasses and ungraduated ruler.



Let AB be the given straight line, and P the given point in AB.

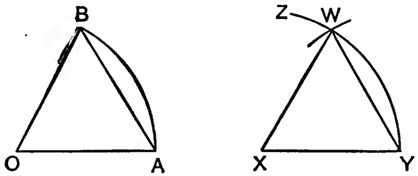
With P as a centre and any convenient radius, draw an arc cutting AB at C and D. With C and D as centres, and any convenient radius longer than CP, draw two arcs cutting one another at X. Draw the straight line XP, which will be the required perpendicular.

5. To construct an arc which shall be equal to a given arc both in degrees and in length.

(*a*) With the aid of a protractor:—

This method is explained on pp. 42 and 159.

(*b*) With the aid of compasses and ungraduated ruler: —



Let AB be the given arc.

If the centre O is not given with the arc, find it by the aid of problem 9 on p. 158. Draw the chord AB and the radius OB.

With any point X as a centre, and a radius equal to OB, draw an arc YZ evidently greater than the required arc.

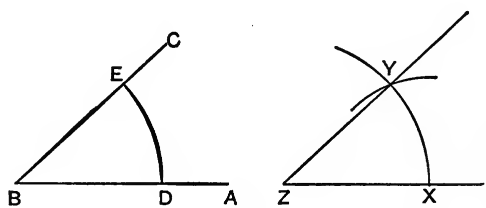
With Y as a centre, and a radius equal to the chord AB, draw an arc cutting YZ at W. YW will be the required arc.

6. To construct an angle which shall be equal to a given angle.

(*a*) With the aid of a protractor: —

This problem is explained on p. 42.

(*b*) With the aid of compasses and ungraduated ruler: —



Let ABC be the given angle.

With B as a centre, and any convenient radius, draw an arc DE between the sides of the angle.

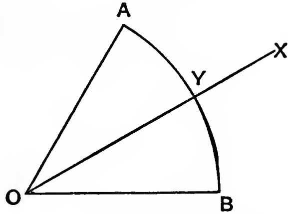
Then construct an arc XY equal to DE.

From Z, the centre from which XY is drawn, draw the radii ZX and ZY.

XZY will be the required angle.

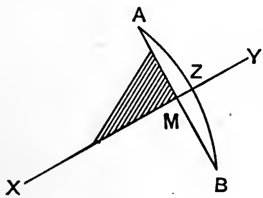
7. To bisect a given arc.

(*a*) With the aid of a protractor and ruler.



Let AB be the given arc, and O the centre of its circle. Draw the radii OA and OB.

Measure the angle AOB with a protractor and divide the result by 2. Draw OX so as to make an angle equal to the quotient, with O as a vertex and OA or OB as one side, and crossing AB at Y. Y will be the middle point of the arc AB.

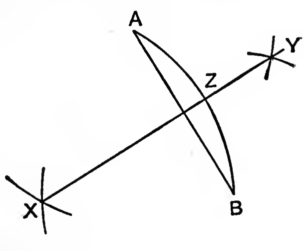


(*b*) With the aid of a square and graduated ruler.

Let AB be the given arc. Draw the chord AB and find its middle point M with a graduated ruler. At M draw XMY perpendicular to the chord AB and crossing the arc AB at Z.

Z will be the middle point of the arc AB.

(*c*) With the aid of compasses and ungraduated ruler.



Let AB be the given arc.

Draw the chord AB and bisect it (as explained on p. 171) with the line XY crossing the arc AB at Z.

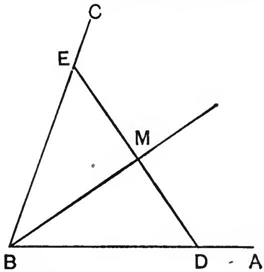
Z will be the middle point of the arc AB.

8. To bisect a given angle.

(*a*) With the aid of a protractor and ruler: —

The method is similar to that for bisecting a given arc.

(*b*) With the aid of a square and graduated ruler: —



Let *ABC* be the given angle.

Measure from *B* any convenient equal distances, *BD* on *BA* and *BE* on *BC*.

Draw the straight line *DE*.

With a square draw a perpendicular *BM* to *DE*.

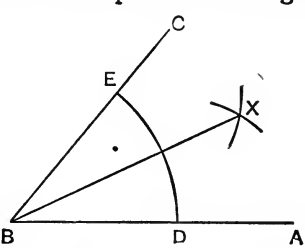
BM will bisect the angle *ABC*.

(*c*) With the aid of a graduated ruler alone: —

Proceed as in case (*b*) until the line *DE* has been drawn.

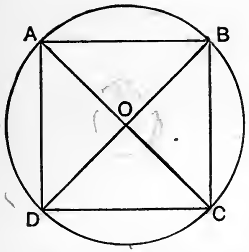
Then bisect *DE* with a graduated ruler, and connect the middle point with a straight line, which will be the same as the line *BM* and will bisect the angle.

(*d*) With the aid of compasses and ungraduated ruler: —



Let *ABC* be the given angle.

With *B* as a centre, and any convenient radius, draw the arc *DE* between the sides of the angle. Bisect this arc with the line *BX*, which will also bisect the angle *ABC*.



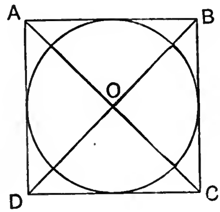
9. To circumscribe a circle about a square.

Let $ABCD$ be a square.

Draw the diagonals, and let O be their point of intersection.

With O as a centre, and a radius equal to OA $(= OB = OC = OD)$, draw a circle which will be the required circle circumscribed about the square.

10. To inscribe a circle in a square.



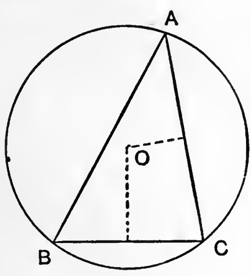
Let $ABCD$ be a square.

Draw the diagonals, and let O be their point of intersection.

With O as a centre, and a radius equal to one-half a side of the square, draw a circle which will be the required circle inscribed in the square.

Construct the following squares, and draw circles in and about each:—

1.	Side 2 cm.	**6.**	Side 1 inch.
2.	" 3 "	**7.**	" 2 inches.
3.	" 4 "	**8.**	" 3 "
4.	" 5 "	**9.**	" $1\frac{1}{2}$ "
5.	" 25 mm.	**10.**	" $2\frac{1}{2}$ "



11. To circumscribe a circle about a triangle.

Let ABC be a triangle of any kind.

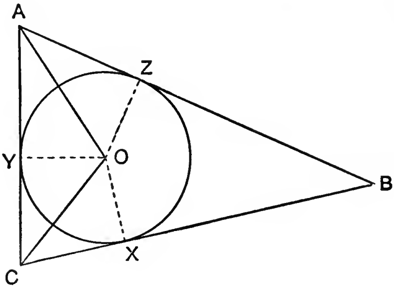
Erect perpendiculars at the middle points of any two sides, and prolong them until they meet at O, which will be equally distant from all three vertices of the triangle.

With O as a centre, and a radius equal to OA $(= OB = OC)$, draw a circle, which will be the required circle circumscribed about the triangle.

12. To inscribe a circle in a triangle.

Let ABC be a triangle of any kind.

Bisect any two of the angles, and prolong the bisectors until they meet at O, which will be equally distant from all three sides.



With O as a centre, and a radius equal to the perpendicular OX ($= OY = OZ$), draw a circle, which will be the required circle inscribed in the triangle.

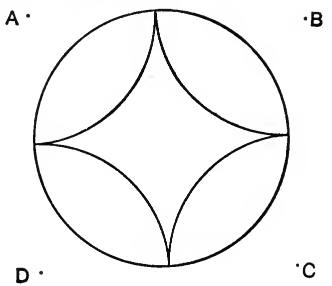
In the case of an *equilateral* triangle the centre of the inscribed circle is the same point as the centre of the circumscribed circle.

Construct equilateral triangles with the following sides, and circumscribe and inscribe a circle in each case:—

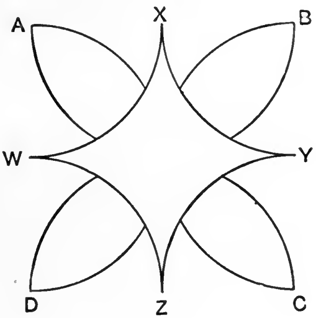
1. Side 3 cm.
2. " 4 "
3. " 5 "
4. " 25 mm.
5. " 35 "
6. Side 2 inches.
7. " 3 "
8. " 4 "
9. " $1\frac{1}{2}$ "
10. " $2\frac{1}{2}$ "

13. Miscellaneous problems on constructions.

1. A, B, C, and D are the vertices of a square circumscribed about a circle. With each vertex as a centre, and a radius equal to the radius of the circle, an arc is drawn within the circle and bounded by its circumference. Let the radius of the circle be one inch.

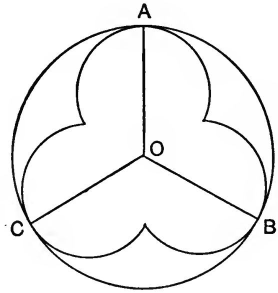


2. Construct a square. Then with each vertex as a centre, and a radius equal to one of the sides, draw an arc within the square and bounded by its sides.
3. Construct a square and draw its diagonals. Then at the middle points of the sides, with a radius equal to one-fourth of a diagonal, draw circles.
4. Construct a square. Then with each vertex as a centre, and a radius equa to one-half a diagonal, draw an arc within the square and bounded by its sides. Connect the ends of these arcs so as to form a regular octagon.
5. *A*, *B*, *C*, *D* are the vertices of a square, and *X*, *Y*, *Z*, *W* are the middle points of its sides. With each of these points as a centre, and a radius equal to one-half a side of the square, arcs are drawn so as to form the annexed figure. Let the side of the square be two inches.



6. Construct a square. Then with the vertices and middle points of the sides as centres, and a radius equal to one-half a side, draw arcs outside the square and bounded by its sides.
7. Construct a square. Then upon each side as a diameter draw a semi-circumference within the square.
8. Construct a square and draw its diagonals. With the vertices as centres, and a radius equal to one-fourth of a diagonal, draw arcs within the square and bounded by its sides. With the point of intersection of the diagonals as a centre, and the same radius as before, draw a circumference, which will be tangent to the other arcs.
9. Construct a circle and determine the vertices of an equilateral inscribed triangle. With each of these points as a centre, and a radius equal to the radius of the circle, draw an arc within the circle and bounded by the circumference.
10. Construct an equilateral triangle. Then with each vertex as a centre, and a radius equal to a side of the triangle, draw an arc between the other two vertices.
11. *A*, *B*, and *C* are the vertices of an inscribed equilateral triangle, and *OA*, *OB*, and *OC* are radii. Upon these radii as diameters arcs are drawn so

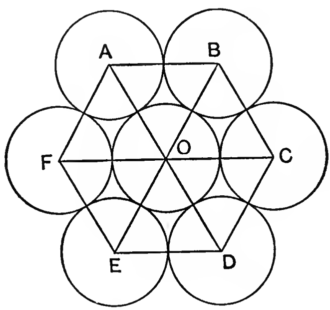
as to meet one another as in the annexed figure. Let the radius of the circle be two centimetres, and construct the figure.



12. Construct an equilateral triangle. Then with each vertex as a centre, and a radius equal to one-half a side of the triangle, describe a circumference. The three circles will be tangent to one another.

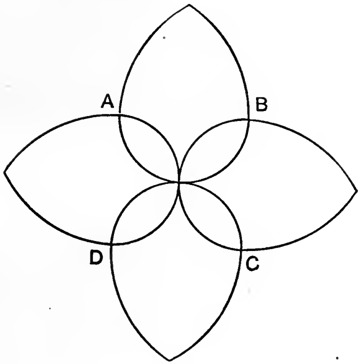
13. Construct a circle and determine the vertices of a regular inscribed hexagon. With these points as centres, and a radius equal to the radius of the circle, draw arcs within the circle and bounded by the circumference.

14. *ABCDEF* is a regular hexagon, whose diagonals cross each other at *O*. With *O* and each vertex of the hexagon as centres, and a radius equal to one-half a side of the hexagon, circles are constructed, six of which are tangent to the seventh. Let the side of the hexagon be one inch, and construct the figure.



15. Construct a circle and inscribe in it a regular hexagon. Upon the sides of the hexagon as diameters construct circles.

16. *A*, *B*, *C*, and *D* are the vertices of a square. Upon the sides of the square as diameters semicircumferences are drawn inwardly. With the vertices of the square as centres, and a radius equal to a side, arcs are drawn out-



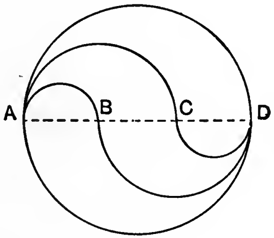
wardly until they meet. Let the side of the square be 3 cm., and construct the figure.

17. Determine the vertices of a square, and construct a figure having the arcs of the preceding problem reversed in position, the semicircumferences being drawn outwardly and the other arcs inwardly.

18. Construct a circle and determine the vertices of a regular inscribed dodecagon. Omitting every third vertex, with the rest of the vertices as centres, and a radius equal to the radius of the circle, draw arcs from the circumference to the centre.

19. Construct an equilateral triangle. At the middle points of the sides draw perpendiculars to meet at a point, and prolong each in the opposite direction so that the part outside may be equal to the part within the triangle. With the outer ends of these perpendiculars as centres, draw arcs outside the triangle with the sides for chords.

20. *AD* is the diameter of a circle and is divided into three equal parts at *B* and *C*. Upon *AB*, *AC*, *BD*, and *CD* as diameters semicircumferences are drawn, two on each side of the diameter. Construct the figure, taking 6 cm. as the diameter of the circle.

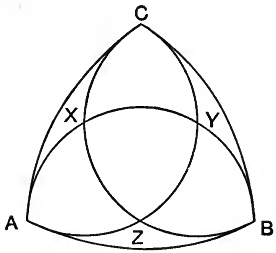


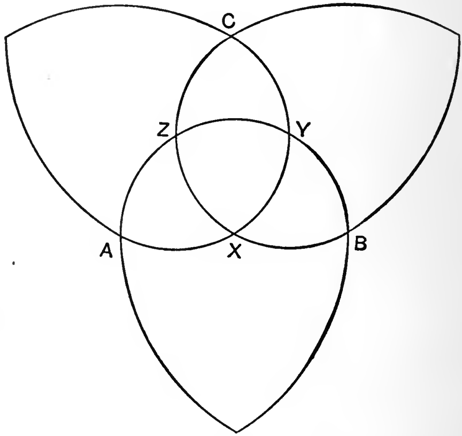
21. Construct a circle with a diameter of 8 cm. divided into four equal parts, and draw four semicircumferences so as to form a figure like that of the preceding problem.

22. Construct a circle and determine the vertices of an inscribed square. With these points as centres, and a radius equal to the radius of the circle, draw semicircumferences which will all pass through the centre of the circle and meet so as to form a four-leaved figure.

23. Determine the vertices of an equilateral triangle. Then with these points as centres, and a radius equal to one-half a side of the triangle, draw circles.

24. Construct a figure similar to the first below. *A*, *B*, and *C* are the vertices of an equilateral triangle; each side should be 4 cm. *X*, *Y*, and *Z* are the middle points of the sides.





25. Construct a figure similar to the second above. *A*, *B*, and *C* are the vertices of an equilateral triangle; *X*, *Y*, and *Z* are the middle points of the sides. Three semicircumferences are drawn upon the sides, and six arcs having the vertices for centres and radii equal to one of the sides. Let the distance from *A* to *B* be $1\frac{1}{2}$ inches.

26. Construct a figure having the same arcs as in problem 25, but reverse their positions, so that the semicircumferences will be drawn outwardly and the other arcs inwardly.

27. Construct a circle and determine the vertices of a regular inscribed octagon. With these points as centres, and a radius equal to a side of the octagon, draw arcs within the circle and bounded by the circumference.

CHAPTER XXVI

AREAS

Areas of Polygons. Review what is said about areas on pp. 13, 14.

FOR REFERENCE

For measurement of areas, two tables are in common use, the Metric and the English.

METRIC TABLE

100 sq. millimetres (sq. mm.) = 1 sq. centimetre (sq. cm.) = $\frac{1}{6}$ sq. inch, nearly.
100 sq. centimetres = 1 sq. decimetre (sq. dcm.) = $\frac{1}{9}$ sq. ft. nearly.
100 sq. decimetres = 1 sq. metre (sq. m.) = 1 centar (ca.) = $1\frac{1}{5}$ sq. yds. nearly.
100 sq. metres = 1 sq. dekametre (sq. dkm.) = 1 ar (a) = 4 sq. rods nearly.
100 sq. dekametres = 1 sq. hektometre (sq. hkm.) = 1 hektar (hka) = $2\frac{1}{2}$ acres nearly.
100 sq. hektometres = 1 sq. kilometre (sq. km.) = $\frac{3}{8}$ sq. mile nearly.

ENGLISH TABLE

144 sq. inches (sq. in.)	= 1 sq. foot (sq. ft.)	= $9\frac{1}{3}$ sq. dcm.	nearly.
9 sq. feet	= 1 sq. yard (sq. yd.)	= $\frac{4}{5}$ sq. m.	"
$30\frac{1}{4}$ sq. yards = $272\frac{1}{4}$ sq. ft.	= 1 sq. rod (sq. rd.)	= $25\frac{1}{3}$ sq. m.	"
160 sq. rods	= 1 acre	= $40\frac{1}{2}$ ars	"
640 acres	= 1 sq. mile	= $2\frac{3}{5}$ sq. km.	"

1. **Area of a Rectangle.** Review the explanation of the area of rectangles on p. 28.

1. A piece of paper is 7 cm. long and 4 cm. wide, having the shape of a rectangle. Draw a plan of the exact size; divide it into square centimetres by drawing lines; then count the squares, writing inside each square its number from one upwards.

2. Draw a rectangle 8 cm. long and 2 cm. 5 mm. wide, and draw lines dividing it into squares and parts of squares. How many sq. centimetres does it contain? Then taking scissors cut up the rectangle into the parts which you marked off, match together the parts of squares, and see how many squares there are altogether. Is the number the same as you found before by calculation?

3. Do the same as in the preceding question with a rectangle 8 inches long and $1\frac{1}{4}$ inches wide. How many of the parts of squares must you put together in order to make a complete square?

4. Do the same as in the preceding question with a rectangle $3\frac{1}{2}$ inches long and $2\frac{1}{2}$ inches wide. In this case one of the squares will be incomplete: how much does it lack to become a complete square?

5. A "shuffle board" is 30 feet long and 20 inches wide, and has the shape of a rectangle. What would you consider to be the most convenient unit in which to calculate its area?

Draw a plan of the shuffle board on the scale of $\frac{1}{40}$; that is, make each edge of your rectangle one-fortieth of the corresponding edge of the shuffle board. Then calculate the area of the board without drawing lines to divide the figure into units. Give the answer both in square feet and in square inches: does the result change or confirm your opinion as to the best unit to use in this case?

6. A "bagatelle board" is 1 metre 5 dcm. long and 7 decimetres wide. Draw a plan of the board on the scale of $\frac{1}{100}$, and calculate the area of the board both in square metres and in square decimetres. How does the area of the board compare with the area of your plan?

7. A cricket field is 70 yards long and 50 yards wide, and has the shape of a rectangle. Draw a plan of the field on the scale of 20 yards to the inch; that is, represent a length of 20 yards on your plan by one inch.

What is the area of your plan?

What is the area of the field?

8. A surveyor finds that a piece of land extends northeast 150 metres, then northwest 60 metres, then southwest 150 metres, and lastly southeast 60 metres. Draw a plan of the land on the scale of $\frac{1}{1000}$, and find the areas of the plan and the land.

9. A tennis court for the single-handed game is a rectangle 27 feet wide and 78 feet long. The net crosses the middle points of the side lines; that is, the two longer sides. The half-court line connects the middle points of the base-lines; that is, the two shorter sides. The two service lines are drawn parallel to the base-lines, and each is 21 feet from the net.

Draw a diagram of the court on the scale of $\frac{1}{216}$; that is, you will represent a length of eighteen feet by one inch on your plan. Then calculate: —

(*a*) The area of your plan.

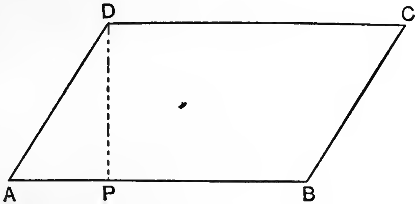
(*b*) The areas of the eight divisions of your plan.

(*c*) The area of the court.

(*d*) The areas of the eight divisions of the court.

10. A tennis court for the four-handed game has the same length as the court for the single-handed game, but is 36 feet wide instead of 27 feet. How much additional area does it contain?

2. **Area of a Parallelogram.** The area of a parallelogram is equal to the product of its base and altitude.

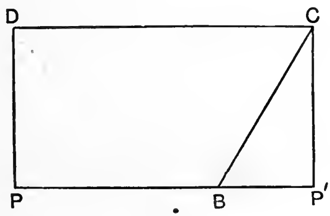


The *base* is any one of the sides.

The *altitude* is the perpendicular distance between the base and the opposite side.

Thus in *ABCD, AB* is the base, and *DP* the altitude.

Draw accurately on paper a parallelogram *ABCD* with any convenient sides and angles. Take *AB* as the base, and draw the altitude *DP*.



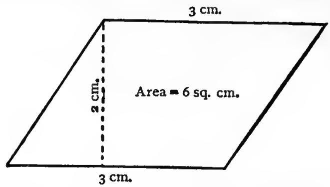
Cut the parallelogram out from the paper. Then cut off the triangle *ADP*, and match it upon the figure in the position *BCP'*.

You can paste the two parts together with a strip of paper on the back.

You have now changed the parallelogram into a rectangle, keeping the same base and altitude.

As the area of the rectangle is the product of its base and altitude, this is also the area of the parallelogram.

Draw the following parallelograms according to the descriptions given, and calculate their areas. Write the values of the given parts and the area in and about the diagrams.

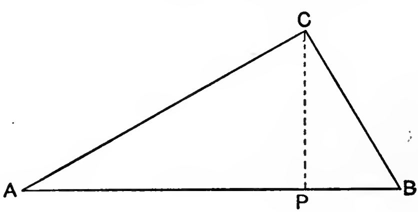


1. Two opposite sides each 3 cm. long, and 2 cm. apart.
2. Two opposite sides each 4 cm. long, and 1 cm. apart.
3. " " " " 3 " " " 3 " "
4. " " " " 2 " " " 4 " "
5. A base 2 cm. 5 mm. long, distant 1 cm. from the opposite side.
6. " " 4 " " " 2 " 4 mm. from the opposite side.
7. " " 1 " 8 " " " 3 " 2 " " " " "
8. Two sides each 8 cm. long, two sides each 4 cm. long, two angles of 45°, and 2 angles of 135°.
9. Two sides each 6 cm. long, two sides each 4 cm. long, two angles of 60°, and two angles of 120°.
10. Two sides of 6 cm. 4 mm. and 4 cm., making with each other an angle of 150°.
11. Sides of 3 and 8 cm., inclined to each other at an angle of 80°.
12. A surveyor marks off a line on the ground running due east, 8 metres long. From the east end he draws a line 5 metres long, running northwest, thus making an angle of 45° with the first line. From the north end of the second line he draws a line due west, 8 metres long. Lastly he draws a line connecting the west ends of the first and third lines.

Draw a plan of the enclosed land (scale $\frac{1}{100}$), and find :—

(*a*) The direction in which the fourth line runs.
(*b*) The angles which the fourth line makes with the first and third.
(*c*) The length of the fourth line as it appears in your plan.
(*a*) The real length of the line on the ground.
(*e*) The class to which the figure belongs.
(*f*) The area of your plan.
(*g*) The real area of the ground.

3. **Area of a Triangle.** The area of a triangle is equal to one-half the product of its base and altitude.

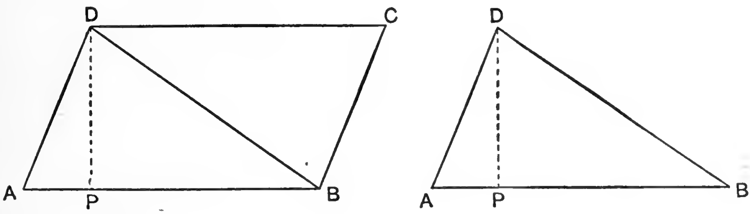


The *base* is any one of its sides.

The *altitude* is the perpendicular distance from the base to the vertex of the opposite angle.

Thus in the triangle ABC, AB is the base, and CP is the altitude.

Draw on paper a parallelogram $ABCD$ of any convenient sides and angles. Taking AB for the base draw the altitude DP. Draw the diagonal DB.



Cut out the parallelogram from the paper, and cut it into two parts along the diagonal DP.

Then turn one part around, place it directly over the other, and you will see that the two are equal, and the triangle is one-half the parallelogram.

Now the area of the parallelogram is the product of its base and altitude. So the area of the triangle, which is one-half the parallelogram, is one-half the product of the base and altitude of the parallelogram; that is, one-half the product of its own base and altitude.

Construct the following triangles and calculate their areas, writing in the values upon the diagrams: —

1. Base 8 cm., altitude 4 cm.
2. " 4 " " 8 "
3. " 5 " " 3 "
4. " 3 " " 5 "
5. A right isosceles triangle whose equal sides are each 5 cm. long.
6. A right triangle, the sides of the right angle being 5 cm. and 3 cm.
7. An isosceles triangle, whose equal angles are each 45°, and whose equal sides are each 5 cm. long.

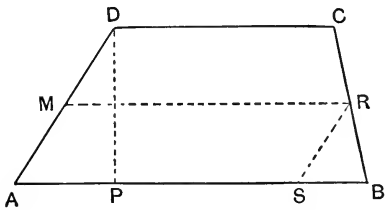
8. Draw a right triangle. Then draw two more lines:—
(*a*) So as to form a rectangle.
(*b*) So as to form a parallelogram.

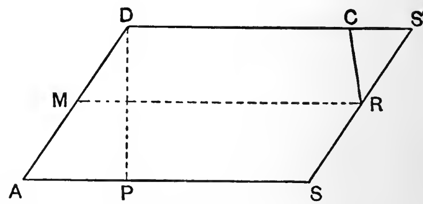
9. Draw an isosceles triangle. Then draw two more lines:—
(*a*) So as to form a rhombus.
(*b*) So as to form a parallelogram.

10. Draw a right isosceles triangle. Then draw two more lines:—
(*a*) So as to form a square.
(*b*) So as to form a rhombus.

4. **Area of a Trapezoid.** The area of a trapezoid is equal to its altitude multiplied by one-half the sum of the parallel sides.

The *altitude* of a trapezoid is the perpendicular distance between the parallel sides.





Thus in the trapezoid $ABCD$, AB and CD are the parallel sides, and DP is the altitude. The parallel sides are also called *bases*.

Draw on paper a trapezoid $ABCD$, in which AB and CD are the parallel sides. Draw MR connecting the middle points of AD and BC. Lay off on AB the distance AS equal to MR. Draw SR.

Cut out the trapezoid from the paper; cut off the triangle SBR and place it in the position $S'CR$; paste the parts together.

You have now changed the trapezoid into a parallelogram.

The two parallel sides of the trapezoid have become equal sides of the parallelogram, and one-half the sum of the parallel sides is equal to AS, the base of the parallelogram.

The altitude DP is unchanged.

Now, since the area of the parallelogram is the product of its altitude and base, the area of the trapezoid is the product of its altitude and one-half the sum of its bases.

Draw the following trapezoids and calculate their areas, writing in the values upon the diagrams: —

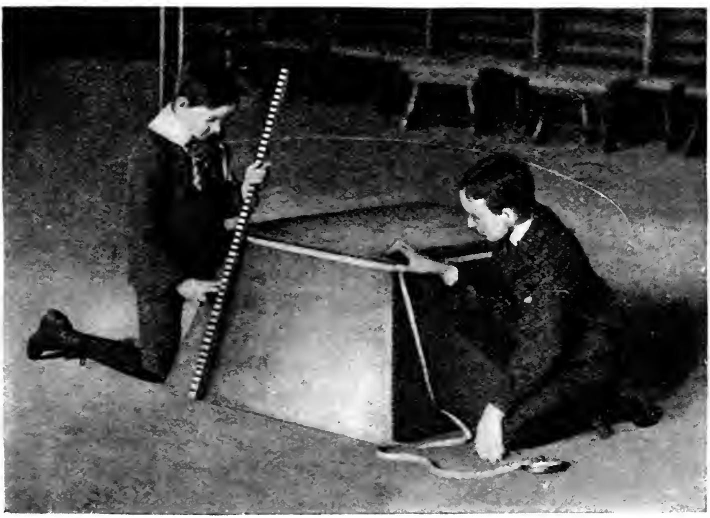
1. Bases 4 cm. and 2 cm.; altitude 3 cm.
2. " 3 cm. 5 mm. and 2 cm. 7 mm.; altitude 2 cm. 4 mm.
3. " 5 " 2 " " 3 " 6 " " 1 "
4. The sides and angles of a trapezoid taken in order are 7 cm., 20°, 1 cm. 8 mm., 160°, 4 cm. 6 mm., 140°, 9 mm., 40°.
5. A face of the pedestal of a statue has the shape of a trapezoid. The parallel sides of the trapezoid are 6 metres and 4 metres long; the other sides are each 2 metres long. The angles at the ends of the longest side are each 60°. Draw a plan of the trapezoid on the scale of $\frac{1}{100}$, and calculate the area of the real figure.
6. A plot of ground has the shape of a trapezoid. The longer base is 48 metres, and makes an angle of 30° with each of the sides which it meets, these being each 24 metres long.

Draw a plan of the ground on the scale of $\frac{1}{100}$, and calculate: —

(*a*) The length of the fourth side on your plan.

(*b*) The area of the ground.

7. Some boys, who are measuring the dimensions of a platform which has the shape of the frustum of a pyramid, find that the slant height is 1 ft. 8 in. on all sides. The perimeter of the lower base is 14 ft. and that of the



" How many square feet of boards are there in the platform? "

upper base is 11 ft. The bases are rectangles, one edge of the upper base being 3 ft.

How many square feet are there in the lateral surface and upper base together?

8. The bank of a reservoir has the shape of a trapezoid. The upper and lower edges are 20 and 30 metres long, and 5 metres apart. The other two edges make each an angle of 45° with the lower edge.

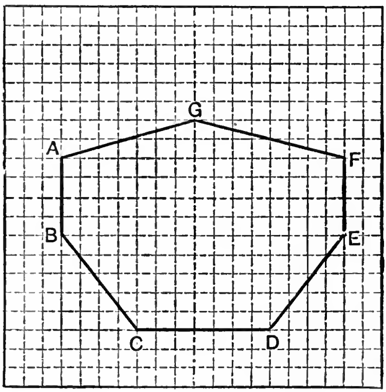
Draw a diagram, naming your own scale, and calculate:—

(*a*) The lengths of the two edges which are not parallel.

(*b*) The angles which those edges make with the upper edge.

(*c*) The area of the reservoir's bank.

5. **Area of a Polygon.** The area of a polygon can be estimated by the aid of transparent paper ruled in little squares whose area has been measured. Lay the paper over the polygon,



count the number of whole squares which are within the perimeter, and estimate the sizes of the parts of squares. The sum of all is the area of the polygon.

The paper in the figure is ruled in squares with an edge $\frac{1}{10}$ of an inch long:—

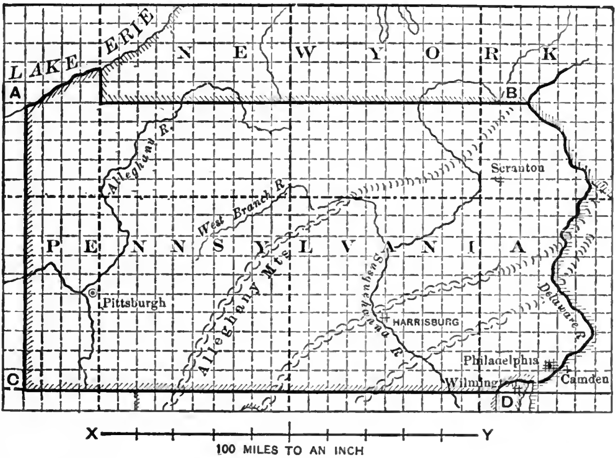
What is the area of one of the squares?

Can you see that *BC* is the diagonal of a certain square?

Can you see that *AG* is the diagonal of a rectangle? If so, you may find exactly what are the sums of the partial squares which adjoin these lines.

What is the area of the whole polygon *ABCDEFG*?

This method is useful in finding roughly the areas of countries, states, townships, etc., from maps. Suppose, for instance, that you wish to find the area of the State of Pennsylvania from a map. First, you must ascertain the scale on which the map is drawn; this is given by the line XY as 100 miles to the inch. If, therefore, your measuring paper is

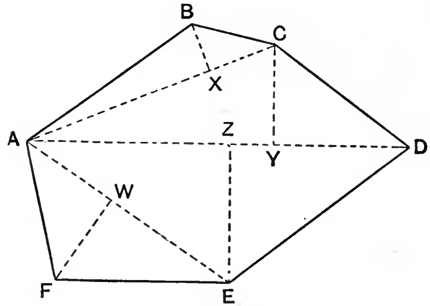


ruled in squares with edges $\frac{1}{10}$ of an inch long, each square represents an area of 100 square miles. Then, applying your measuring paper, you will notice that the part of the map which is indicated by the letters $ABCD$ is a rectangle enclosing 25×15 or 375 of the squares. The rest of the area is covered by squares and parts of squares which together make about 75 more, and added to the others give 450 for the total number. Since each square represents an area of 100 square miles, the area of Pennsylvania is about 45,000 square miles.

There are several methods of calculating the area of a polygon with greater accuracy.

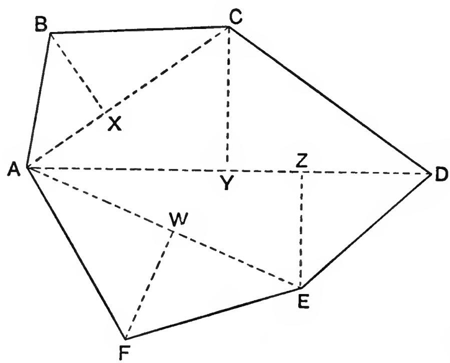
1st. The polygon can be divided into triangles, whose areas are found separately and then added.

The bases of the triangles are diagonals drawn from one of the vertices of the polygon; the altitudes are perpendiculars drawn to the diagonals from the opposite vertices of the triangles.



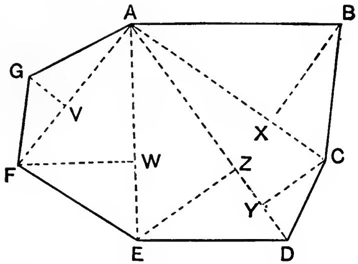
1. Calculate the area of the polygon $ABCDEF$ from the following measurements: —

$AC = 11$ rods; $AD = 16$ rods; $AE = 11$ rods; $BX = 2$ rods; $CY = 4$ rods; $EZ = 6$ rods; $FW = 4$ rods.



2. Calculate the area of the polygon $ABCDEF$ from the following measurements: —

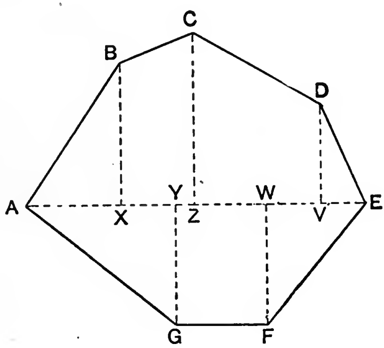
$AC = 10$ metres; $AD = 17$ metres; $AE = 13$ metres; $BX = 4$ metres; $CY = 6$ metres; $EZ = 6$ metres; $FW = 5$ metres.



3. Calculate the area of the polygon $ABCDEFG$ from the following measurements: —

$AC = 10$ metres; $AD = 11$ metres; $AE = 9$ metres; $AF = 8$ metres; $BX = 5$ metres; $CY = 3$ metres; $EZ = 5$ metres; $FW = 5$ metres; $GV = 2$ metres.

2d. The area of a polygon can be found by drawing its longest diagonal and perpendiculars upon this diagonal from the vertices. The polygon is thus divided into trapezoids, rectangles, or right triangles, whose areas are found separately and then added.



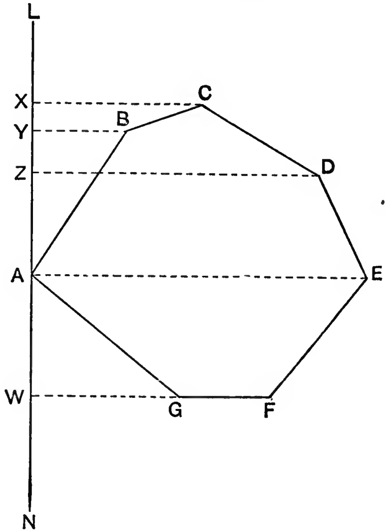
Calculate the area of the polygon $ABCDEFG$ from the following measurements: —

$BX = 6$ metres; $CZ = 7$ metres; $DV = 4$ metres; $GY = 5$ metres; $FW = 5$ metres.

$AX = 4$ metres; $XY = 2$ metres; $YZ = 1$ metre; $ZW = 3$ metres; $WV = 2$ metres; $VE = 2$ metres.

3d. The area of a polygon can be found by a method commonly used by land surveyors.

A line, LN, called "the base line," is drawn at one of the vertices, and perpendiculars are drawn to this from the other vertices, forming trapezoids, rectangles, or right triangles,



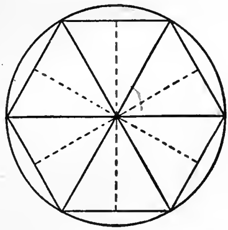
whose areas are found separately and added. Then from the sum there is subtracted the areas of the parts which lie outside the polygon. In this figure the base line LN is drawn perpendicular to the diagonal AE. The parts to be subtracted from the whole area consist of a trapezoid and two right triangles.

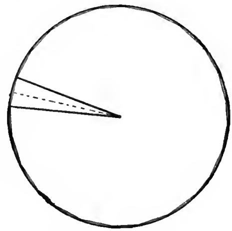
Calculate the area of the polygon $ABCDEFG$ from the following measurements: —

$CX = 7$ metres; $BY = 4$ metres; $DZ = 12$ metres; $EA = 14$ metres; $FW = 10$ metres; $GW = 6$ metres; $XY = 1$ metre; $YZ = 2$ metres; $ZA = 4$ metres; $AW = 5$ metres.

Compare the result with that of the preceding problem, as the two polygons are the same.

6. **Area of a Circle.** The area of a circle can be found by calculating the length of the circumference, multiplying that by the length of the radius, and dividing the product by 2.





This rule depends upon regarding the area of the circle as equal to the sum of the areas of a number of equal isosceles triangles whose bases are chords and whose vertices opposite to the chords meet at the centre of the circle.

If there were only six of these triangles, composing a hexagon, as in figure 1, there would be considerable difference between the area of the circle and that of the polygon. But if the number of triangles were increased only to twenty-four, as in figure 2, the area of the polygon would approach much nearer the area of the circle. It is to be noticed also that the altitudes of the triangles in figure 2 are each nearly equal to the radius of the circle; and the sum of the bases is nearly equal to the circumference of the circle. If the number of triangles were further increased, they would form a polygon which could hardly be distinguished from the circle, though there would always be a difference.

Now the sum of the areas of the triangles can be found by multiplying the sum of their bases by their altitude, and dividing the product by 2.

So the area of the circle can be found by multiplying its circumference by its radius, and dividing the product by 2.

Suppose the radius of a circle to be 4 cm.

Then the circumference $= 2 \times 3\frac{1}{7} \times 4 = 25\frac{1}{7}$ cm.

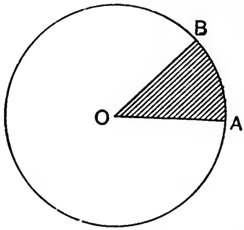
And the area $= 4 \times 25\frac{1}{7} \div 2 = 50\frac{2}{7}$ sq. cm.

Calculate the areas of the following circles, considering π to be equal to $3\frac{1}{7}$: —

1.	Radius = 7 cm.	**6.**	Diameter = 10 cm.
2.	" = 3 "	**7.**	" = 2 cm.
3.	" = 14 "	**8.**	" = 12 "
4.	" = 1 inch.	**9.**	" = 4 inches.
5.	" = 2 inches.	**10.**	" = 8 inches.

7. **Sector.** A *sector* is a part of a circle included by two radii and an arc, as AOB.

A sector is often described by the size of the angle between the radii; thus if the angle AOB is $45°$, the sector is called "a sector of $45°$."



A sector of any required size is constructed by drawing two radii forming an angle of the size indicated.

Construct the following sectors.

1.	$45°$, radius	2 cm.		6.	$30°$, radius	1 inch.	
2.	$120°$, "	3 "		7.	$60°$, "	2 inches.	
3.	$90°$, "	4 "		8.	$45°$, "	$1\frac{1}{2}$ "	
4.	$100°$, "	2 "		9.	$90°$, "	2 "	
5.	$20°$, "	4 "		10.	$120°$, "	$1\frac{1}{4}$ "	

To find the area of a sector: —

(*a*) When the length of the radius and the length of the arc are known.

Like the whole circle, the sector can be regarded as composed of a countless number of triangles whose altitude is the radius and the sum of whose bases is the arc.

The area of the sector, therefore, is found by multiplying the length of the arc by the length of the radius, and dividing the product by 2.

Thus, if the radius is $1\frac{1}{2}$ cm. and the arc 2 cm., the area of the sector will be $\frac{3}{2} \times 2 \div 2 = \frac{3}{2}$ sq. cm.

(*b*) When the length of the radius and the angle of the sector are known.

Let the radius be $1\frac{1}{2}$ cm., and the angle $50°$.

The sector is $\frac{50}{360}$ or $\frac{5}{36}$ of the whole circle.

The area of the circle is $\frac{9}{4}\pi$ or $7\frac{1}{14}$.

$\therefore$ The area of the sector is $\frac{5}{36} \times 7\frac{1}{14} = \frac{5}{36} \times \frac{99}{14} = \frac{55}{56}$ sq. cm.

(*c*) When the length of the radius and the number of degrees in the arc are known.

Since the arc and the angle formed by the radii have the same number of degrees, the method of finding the area is the same as in case (*b*).

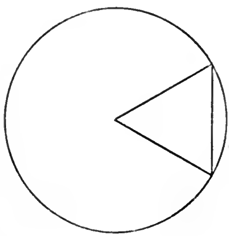
Calculate the areas of the following sectors: —

11.	Radius	4 cm.,	arc 3 cm.		**16.**	Radius	2 cm.,	angle	30°.
12.	"	5 "	" 8 "		**17.**	"	4 "	"	45°.
13.	"	2 "	" 1 "		**18.**	"	7 "	arc	90°.
14.	"	3 "	" 5 "		**19.**	"	3 "	"	100°.
15.	"	7 "	angle 60°.		**20.**	"	1 "	"	120°.

8. **Segment.** A *segment* of a circle is a part of the circle enclosed between a chord and its arc.

The word segment means "a part cut off."

The size is often designated by the number of degrees in its arc; thus if the arc is 60°, the segment is called "a segment of 60°."



The area of a segment can be found by drawing radii to the ends of the arc and deducting the area of the triangle from the area of the sector thus formed.

Construct the following segments: —

1.	Radius	2 cm.,	arc	80°.	**4.**	Radius	1 inch,	arc	90°.
2.	"	3 "	"	90°.	**5.**	"	$1\frac{1}{2}$ inches	"	75°.
3.	"	25 mm.	"	120°.					

Calculate the areas of the following segments: —

6.	Radius	2 cm.,	arc	90°.	**9.**	Radius	1 inch,	arc	90°.
7.	"	3 "	"	90°.	**10.**	Diameter	4 inches,	"	90°.
8.	Diameter	4 "	"	90°.					

9. **Surface of a Sphere.** The surface of a sphere is exactly equivalent to the areas of four circles of the same diameter as the sphere (see p. 113).

1. What is the surface of a sphere whose diameter is 7 cm.?
2. What is the surface of a sphere whose radius is 5 cm.?
3. The diameter of the moon is about 2160 miles. How many square miles are there on its surface?
4. How much would it cost at 2 cents a square foot to paint a hemispherical dome whose diameter is 44 ft.?
5. What is the surface of the greatest globe which can be cut from a cubical block of wood whose edge is 1 dcm.?
6. What is the diameter of a sphere whose circumference is 22 cm.?
7. How many square inches of leather will it take to cover a ball whose circumference is 9 inches?
8. How many balls each 5 cm. in diameter can be covered from a square metre of cloth?
9. How does the surface of a sphere compare with that of a cube whose edge is equal to a diameter of the sphere?
10. How does the surface of a sphere compare with the lateral surface of a cylinder which will exactly contain the sphere?

CHAPTER XXVII

VOLUMES

Volume. Review what is said about volume on p. 13.

(FOR REFERENCE)

For measurement of volumes two tables are in common use, the Metric and the English.

METRIC TABLE

1000 cubic millimetres (cu. mm.)	= 1 cubic centimetre (cu. cm.)		= $\frac{3}{50}$ cu. in. nearly.
1000 cu. centimetres	= 1 cu. decimetre (cu. dcm.)	= 1 litre	= $\frac{1}{30}$ cu. ft. "
1000 cu. decimetres	= 1 cu. metre (cu. m.)	= 1 ster	= $1\frac{3}{10}$ cu. yd. "

ENGLISH TABLE

1728 cubic inches	= 1 cubic foot (cu. ft.)	= 28.3 cu. dcm.	nearly.
27 cu. feet	= 1 cu. yd. (cu. yd.)	= 0.76 cu. m.	"
128 cu. feet	= 1 cord (cd.)	= 3.6 sters	"

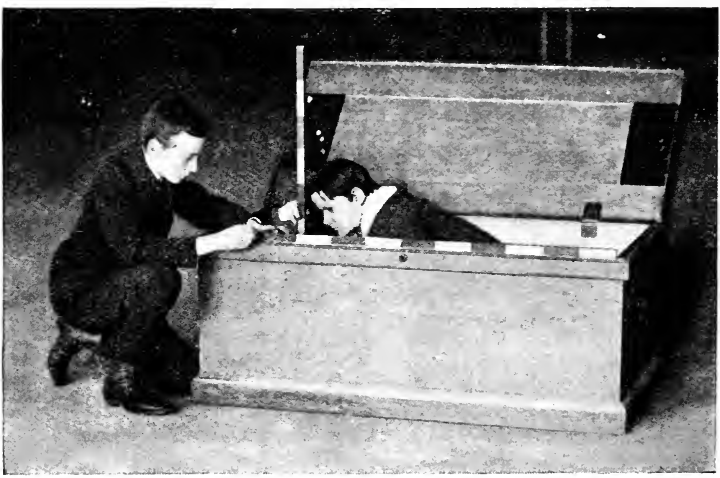
1. **Volume of the Cube.** Review what is said about the volume of the cube on pp. 15–16.

1. What is the volume of a cube whose edge is 5 cm.?
2. How many cubes with an edge of 2 cm. could be cut from a cube whose edge is 10 cm. long?
3. Would it require the same amount of paper to cover the surfaces of the original cube as it would to cover the smaller ones mentioned in the preceding question? If not, how much more paper would be needed in the one case than in the other?
4. How many cubes with an edge of 2 inches can be covered with a sheet of paper 2 feet square?
5. If you had a cubical block of wood with edges 11 inches long, and wished to cut from it as many cubes as possible with an edge of 3 cm., and to use the rest of the block for cubes with an edge of 2 cm., how many of each kind would you get, allowing nothing for waste?

6. In the preceding case, if you were to begin by cutting out all the cubes you could with an edge of 2 cm., how many would you get? If you were then to use the rest of the block for cubes of the largest size possible, what would be the length of their edges, and how many of them would you have?
7. Which will hold the more, five cubical boxes with six-inch edges, or six cubical boxes with five-inch edges?
8. If you had two cubes with edges four inches long, and six cubes with edges two inches long, how many more of the smaller size would you need so that you could form a cube with an edge of six inches by placing them all together?
9. If you have a cubical box whose interior dimensions are each 23 inches, and wish to fill it as completely as possible with cubes of one size, having either a three-inch or a four-inch edge, which kind will leave the least vacant space?

2. **Volume of the Parallelopiped.** Review what is said about the volume of the parallelopiped on pp. 29–30.

1. How many cubic decimetres are there in a chest which is 1 m. 1 dcm. long, 3 dcm. wide, and 4 dcm. 5 cm. deep?



"How much will the chest hold?"

2. If the diagram on p. 19 were folded up so as to form a parallelopiped, what would be its volume?
3. How much paper would be needed to cover a parallelopiped 6 cm. $\times$ 3 cm. $\times$ 2 cm.?
4. Can bricks 8 in. $\times$ 4 in. $\times$ 2 in. be piled up so as to form a cube with an edge of 2 feet? If so, how many would it take?
5. How many cubes with an edge of 5 cm. can be cast from a block of iron 25 cm. $\times$ 15 cm. $\times$ 8 cm.?
6. If the cubes mentioned in the preceding question were to be cut from a block of wood of the same size as the iron, some of the material being necessarily unused, how many cubes could be obtained?
7. How many bricks 8 in. $\times$ 4 in. $\times$ 2 in. would be needed for a wall 80 ft. long, 6 ft. high, and 8 in. thick?
8. If the wall mentioned in the preceding question belonged to a building 30 ft. wide, how many bricks would be needed for all four walls?
9. If you had a block of wood 18 cm. $\times$ 12 cm. $\times$ 8 cm., and wished to cut it either into cubes with an edge of 3 cm., or into parallelopipeds 6 cm. $\times$ 4 cm. $\times$ 2 cm., which would you choose so as to lose the least of the material?
10. Which will hold the more, a certain number of boxes each 7 in. $\times$ 5 in. $\times$ 3 in., or half as many boxes each 14 in. $\times$ 10 in. $\times$ 6 in.?
11. How many cubic decimetres are there in a chest which is 1 m. 2 dcm. 5 cm. long, 3 dcm. 5 cm. wide, and 44 cm. deep?

3. **Volume of the Prism.** Review the subject of prisms, on p. 34. The prism there described has a triangular base, equal to one-half of the square face of a cube, and a height equal to an edge of the cube. The volume of this prism is evidently equal to one-half that of the cube; that is, the volume is equal to the area of its triangular base multiplied by its height.

The same is true of any prism: the volume is equal to the area of the base multiplied by the height. The base is a polygon, whose area can be found by the methods given on pp. 190–194; if the prism is a right prism, the faces are all rectangles, and the height of the prism is equal to the length of the lateral edges.

1. If the diagram on p. 33 were folded up so as to form a prism, what would be the volume?
2. The prism described on p. 123 has for its base a pentagon whose area is about 10.75 sq. cm.

What is the volume of this prism?

What is the total area of its surface?

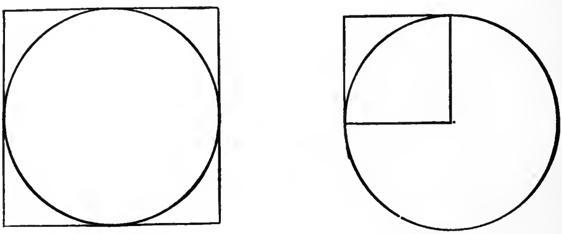
3. Find the volume and the area of the entire surface of a right hexagonal prism, each of whose edges is 5 cm. long, the area of the base being 65 sq. cm.

4. Find the volume and the area of the entire surface of a right prism whose height is 10 cm., and whose base is a right isosceles triangle with the equal edges 5 cm. and the longest edge 7.1 cm.

5. Find the lateral area, the entire area, and the volume of a right prism whose lateral edges are 8 inches long, and whose base is an equilateral triangle with an edge of 2 inches.

4. **Volume of a Cylinder.** Review the experiment on the volume of a cylinder, p. 100.

The cylinder there described has for its base a circle whose diameter is equal to an edge of the cube with which the cylinder is compared; and for its height an edge of the cube. The volume of the cylinder is found to be about three-fourths of the volume of the cube.



The area of the base of the cylinder is about three-fourths of the base of the cube: that is, the circle is equal to about three-fourths of a square constructed on its own diameter; or, since the square on the diameter is four times as great as the square on the radius, the area of the circle is about three times as great as the square on its radius.

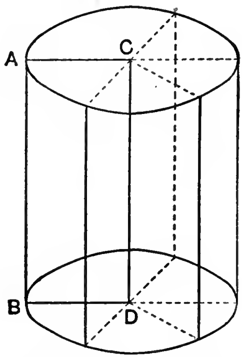
With more accurate measurements the area of the circle would be found to be nearly equal to $\frac{11}{14}$ of the area of the square on its diameter, or $\frac{22}{7}$ of the square on its radius.

There are, then, four expressions which you may use for the area of a circle: —

1. $\frac{3}{4}$ of the square on the diameter.
2. 3 times the square on the radius.
3. $\frac{11}{14}$ of the square on the diameter.
4. $\frac{22}{7}$ of the square on the radius.

The first two are sufficient for rough estimates; the last two are accurate enough for any calculations which you will need to make in elementary geometry.

The volume of any cylinder is equal to the area of its base multiplied by its height.

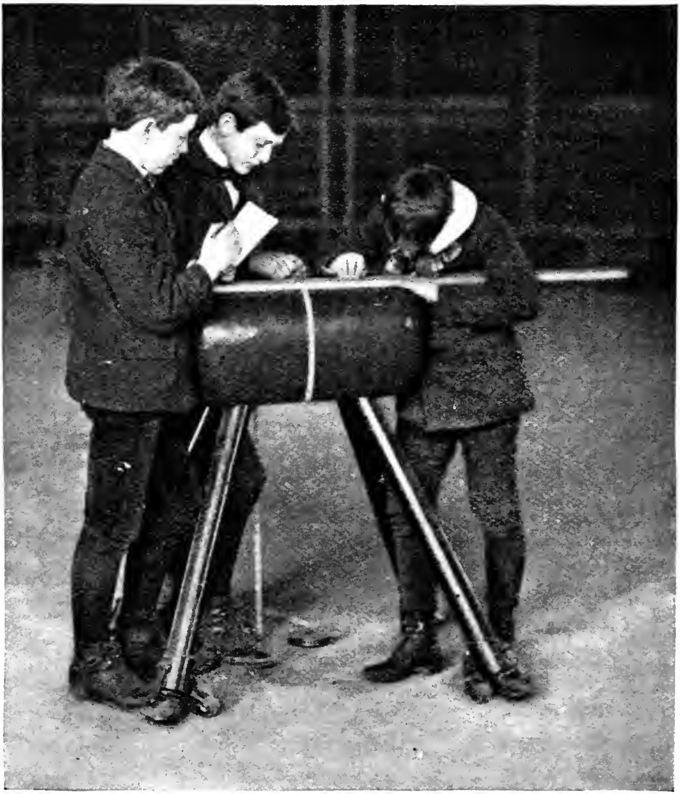


If the straight line which joins the centres of the bases of a cylinder is perpendicular to the bases, the cylinder is called a *right cylinder*.

In this case the lateral (or curved) surface, as was shown on p. 98, is formed by a rectangle having for its edges the circumference of the base and the height of the cylinder. The area of the lateral surface, therefore, is equal to the circumference of the base multiplied by the height of the cylinder.

The length of the circumference of a circle may be considered to be either 3 times or $3\frac{1}{7}$ times the length of the diameter, according to the degree of accuracy required.

1. Find in cubic cm. the volume of the cylinder described on p. 98, as having a diameter and height of 5 cm., obtaining a more accurate value for the area of the base.
2. The radius of the base of a cylinder is 14 cm., and its height is 10 cm. Find, first roughly, and then more accurately: —
   (*a*) The area of the base.
   (*b*) The area of the lateral surface.
   (*c*) The area of the entire surface.
   (*d*) The volume of the cylinder.
3. If you had a block of wood of the dimensions of the right parallelopiped described on p. 19 (4 in. × 3 in. × 2 in.), what would be the volumes,



"How much leather will it take to cover the vaulting-horse?"

by rough estimate, of the greatest cylinders which you could cut from it, using for a base, —

(*a*) The greatest face of the block.

(*b*) The second greatest face of the block.

(*c*) The smallest face of the block.

4. What would it cost to paint the surfaces of the three cylinders mentioned in the previous question at one cent per sq. inch?

5. If you have a cubical box whose interior dimensions are 10 inches, how many cylinders can you pack into it, each having a two-inch diameter and being 4 inches tall?

How much sawdust would you then need, to fill up the vacant space in the box?

6. A party of boys have a metre stick and an English tape measure with which to find the surface and volume of a gymnasium "vaulting-horse" which has the shape of a cylinder with hemispherical ends. They find that the length, not including the ends, is 5 decimetres, and the circumference is 33 inches.

(*a*) What is the entire surface in sq. cm.?

(*b*) " " " " " " sq. ft.?

(*c*) " " " volume in cu. cm.?

(*d*) " " " " " cu. ft.?

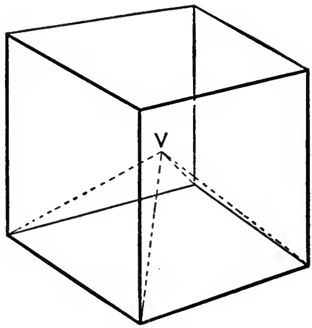
5. **Volume of a Pyramid.** — Review the experiment on the volume of a pyramid, pp. 63–64.

The pyramid there described has a square base, equal to the base of the cube with which the pyramid is compared, and a height equal to the height of the cube. The volume of the pyramid appears by experiment to be one-third of the volume of the cube.

Let us now investigate the volume of the pyramid by a different method. Suppose a pyramid to lie within a cube, the base of the pyramid being one of the faces of the cube, and the apex V being at the centre of the cube; the height of the pyramid, therefore, is equal to one-half the height of the cube. Now if you imagine each face of the cube to be the base of a pyramid having its vertex at V, you will see that the six pyramids just fill the cube. The volume of each pyramid is one-sixth of the volume of the cube; or one-sixth of the area of its base multiplied by the height of the cube; or one-third of the area of its base multiplied by its own height.

The volumes of all pyramids are found by this rule: multiply the area of the base by the height of the pyramid, and divide the product by 3.

The area of the base can be found by the methods you have used in finding the areas of polygons. The height may be measured by resting a ruler horizontally on the apex of the pyramid and some object which has a vertical surface, and then measuring the height along that surface.



If the base is a regular polygon, and the apex lies directly over the centre of the base, the figure is called a *regular* pyramid. The lateral surface, in this case, consists of equal triangles having for bases the edges of the polygon, and for altitudes the slant height of the pyramid.

1. What is the volume of a pyramid, the area of whose base is 24 sq. cm., and whose height is 7 cm.?

2. In the pyramid described on p. 65, the edges are each 5 cm. long; the altitudes of the faces are about 4.3 cm.; the height of the pyramid is about 4.1 cm.
What is the volume of this pyramid?
What is the area of its entire surface?

3. What is the height of one of the six equal pyramids which will exactly fill a cubical box whose interior dimensions are 14 inches?

4. What is the volume of the smallest cubical box within which you could enclose the pyramid whose surface is represented by the diagram on p. 59?

5. What is the volume of a pyramid which could be enclosed in the triangular prism whose surface is represented by the diagram on p. 33, the base of the

pyramid covering the base of the prism, and the apex of the pyramid just reaching the top of the prism?

6. If the rectangular parallelopiped described on p. 19, whose dimensions are 4. in. × 3 in. × 2 in., were divided into six pyramids of three different sizes, each pyramid having one of the faces of the parallelopiped for its base, where would the common point of their apexes be situated?

What would be the volume of each of the six pyramids?

7. The greatest pyramid in Egypt has a base 693 ft. square and a height of 500 ft. What is its volume?

6. **Volume of a Cone.** Review the experiment on the volume of the cone, pp. 106–107.

The cone there described has a base and height equal to the base and height of the cylinder with which it is compared. The volume of the cone is found to be one-third of the volume of the cylinder.

The volume of any cone may be found by multiplying the area of its base by its height and dividing the product by 3.

If a line which joins the apex of the cone with the centre of the base is perpendicular to the base, the figure is called a *right cone.* The lateral (or curved) surface is then formed from a sector whose area is equal to one-half the slant height of the cone multiplied by the circumference of the base.

In the case of a cone which you construct from a diagram of its surface, the angle of the sector will be given with the diagram; but you can calculate this angle directly from a completed model. You will notice that the arc of the sector has the same length as the whole circumference of the base, which it joins. Now if an arc of one circle has the same length as the whole circumference of another circle, their radii must be different; the arc will be the same part of its own circumference as the shorter radius is of the longer one; and the number of degrees in the arc is the same as the number of degrees in the angle of the sector. Thus in the diagram on p. 103, if the radius of the arc is $2\frac{1}{4}$ in., and the radius of the base is 1 in., $1 \div 2\frac{1}{4} = \frac{4}{9}$; and $\frac{4}{9}$ of 360° is 160°, which is the angle of the sector.

If you test the same diagram with the metric measurements there given, you will find that the angle of the sector appears to be 161° instead of 160°; this is because the radius of the sector, by exact calculation, is $5.6\frac{1}{4}$ cm. instead of 5 6 cm.

1. What is the volume of a cone whose height is 10 cm., and the diameter of whose base is 7 cm.?
2. What is the volume of the greatest cone which can be cut from a cubical block of wood whose edge is 10 cm. long?
3. The radius of the base of a cone is 3 in., its height is 4 in., and its slant height is 5 in.

   Find (*a*) The area of the base.
   (*b*) The area of the lateral surface.
   (*c*) The area of the entire surface.
   (*d*) The volume.
   (*e*) The angle of the sector which formed the lateral surface.
4. How many cones can be cast from a cylindrical iron bar 20 inches long and 4 inches in diameter, the cones to be 5 inches tall and to have a diameter of 2 inches?
5. The height of a cone is 12 cm., the slant height 13 cm., and the radius of the base 5 cm.

   Find (*a*) The area of the base.
   (*b*) The area of the lateral surface.
   (*c*) The area of the entire surface.
   (*d*) The volume.
   (*e*) The angle of the sector which formed the lateral surface.
6. Suppose a right triangle, with edges 6, 8, and 10 inches, to revolve, first with the shortest edge, and then with the next shortest edge, for an axis. Find and compare the volumes and the entire surfaces of the two cones thus generated.

7. **Volume of a Sphere.** The volume of a sphere is nearly equal to one-half the volume of a cube whose edge is a diameter of the sphere (see p. 114). A more accurate value can be found by multiplying the volume of the cube by $\frac{11}{21}$.

1. What is the volume of a sphere whose diameter is 7 cm.?
2. What is the volume of a sphere whose radius is 5 cm.?
3. How many cubic miles does the earth contain, the diameter being 7912 miles?
4. If a cubic inch of iron weighs 7 ounces, what is the weight of an iron ball 4 inches in diameter?
5. Eight spherical glass globes, each having a diameter of 6 cm., are to be packed in a cubical box whose edge is 12 cm. How much sawdust will be needed to fill the vacant space?
6. The diameter of the ball on St. Paul's Cathedral is 6 ft. How many cubic feet are there in the contents?
7. How many lead bullets 1 cm. in diameter can be cast from a cylinder of lead whose length is 14 cm. and diameter 35 mm.?
8. If a spherical lump of putty whose diameter is 8 cm. were moulded into a cone of the same diameter, what would be the height of the cone?
9. What is the diameter of a sphere whose volume in cubic inches is the same as the area of its surface in square inches?

10. If a cylindrical box has a diameter equal to its depth, what part of the space will be filled by the greatest sphere which will fit into the box?

8. **Volumes of Irregular Figures.** The volumes of irregular figures may be found by experimenting. For instance, you may take a jar whose volume can be measured, and partly fill it with water, noticing the level at which the water stands. Then if you immerse the irregular-shaped object in the water and notice the new level to which it rises, you can calculate the volume of the object thus indirectly: the apparent increase in the volume of the water will be the volume of the object.

1. In a cylindrical well whose diameter is 4 feet, the water stands 12 feet below the brim; but when a heap of stones are thrown in, the level of the water rises to 8 feet below the brim. What is the volume of the stones?

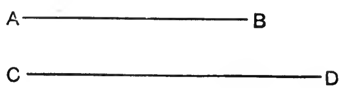
2. A statuette is packed in sawdust in a cubical box whose interior dimensions are 3 dcm., and the box is exactly full; but when the statuette is taken out, the level of the sawdust sinks 12 cm. below the top of the box. What is the volume of the statuette?

CHAPTER XXVIII

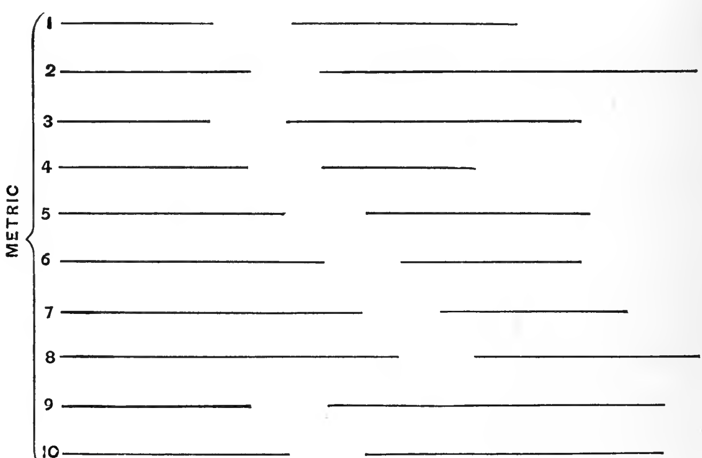
RATIO AND PROPORTION

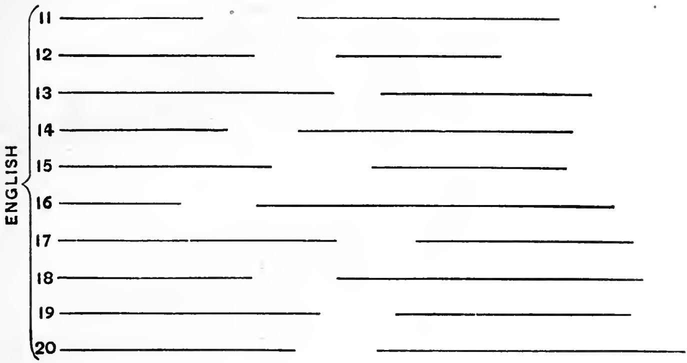
1. A *ratio* is the relation which two things of the same kind have to each other in respect to size; the word means "a reckoning."

For example, if the line *AB* is 3 cm. long, and *CD* is 4 cm. long, the ratio of *AB* to *CD* is $\frac{3}{4}$.

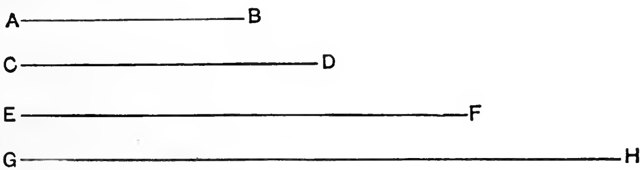


Measure the following lines, and find the ratio of the first to the second in each case:—





2. When two ratios are equal to each other, they are said to form a *proportion*.



For example, if four lines, AB, CD, EF, and GH are 3, 4, 6, and 8 cm. long, so that the ratio of AB to CD is $\frac{3}{4}$, and the ratio of EF to GH is $\frac{6}{8}$, then since $\frac{3}{4}$ is equal to $\frac{6}{8}$, the ratio of the first two lines is equal to the ratio of the last two, and the lengths of the four lines form a proportion.

The proportion is written in this form: —

$$AB : CD = EF : GH$$

which means that AB has the same relation to CD as EF has to GH, or, as it is commonly expressed, "AB is to CD as EF is to GH."

Again, suppose that two squares have edges 2 and 3 cm. long; then the perimeters are 8 and 12 cm. long, and we can say that the perimeters are proportional to two edges; for $8 : 12 = 2 : 3$.

Write in numbers the proportion which exists between the perimeters and two sides of the following polygons: —

1. Two squares whose sides are 1 cm. and 3 cm.
2. Two squares whose sides are 3 cm. and 5 cm.
3. Two squares whose perimeters are 8 cm. and 12 cm.

4. Two equilateral triangles whose sides are 5 cm. and 2 cm.
5. Two rhombuses whose sides are 1 cm. and 4 cm.
6. Two squares whose perimeters are 16 cm. and 12 cm.
7. Two equilateral triangles whose perimeters are 3 cm. and 12 cm.
8. Two equilateral pentagons whose sides are 2 cm. and 3 cm.
9. Two equilateral hexagons whose perimeters are 16 cm. and 20 cm.
10. Two regular decagons whose perimeters are 15 cm. and 20 cm.
11. Two squares whose sides are 3 in. and 4 in.
12. Two squares whose sides are 1 in. and 3 in.
13. Two squares whose perimeters are 8 in. and 12 in.
14. Two equilateral triangles whose sides are 4 in. and 5 in.
15. Two rhombuses whose perimeters are 12 in. and 16 in.
16. Two equilateral pentagons whose sides are 1 in. and 2 in.
17. Two equilateral hexagons whose perimeters are 12 in. and 18 in.
18. Two equilateral triangles whose perimeters are 12 in. and 18 in.
19. Two rhombuses whose sides are 2 in. and 3 in.
20. Two squares whose sides are 2 in. and 3 in.

Draw four lines whose lengths shall form the following proportions: —

21. $2:5=6:15$.
22. $1:2=3:6$.
23. $3:2=6:4$.
24. $2:3=4:6$.
25. $6:2=3:1$.

Notice that in these proportions the product of the two outer numbers, called the *extremes*, is equal to the product of the two inner numbers, called the *means*; thus $2\times 15=5\times 6$; $1\times 6=2\times 3$, etc.

This is expressed by saying that "in every proportion the product of the extremes is equal to the product of the means." By means of this rule, if any three of the numbers which form a proportion are given, the fourth can be found.

Suppose, for example, you have the proportion

$$3:9=2:x,$$

where the fourth number is missing, but is denoted by x. Then, by the rule, $3\times x=18$, or $x=6$; and the proportion can be completed by replacing x by 6 thus: $3:9=2:6$.

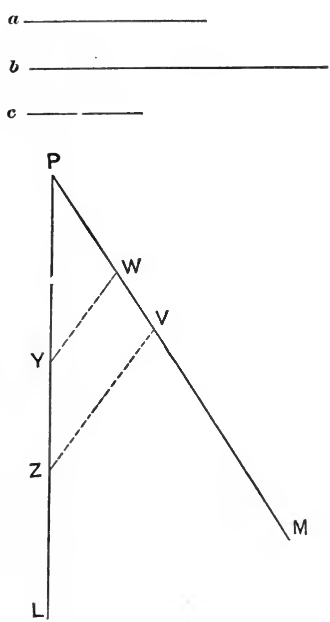
Supply the missing number in the following proportions: —

26. $5:3=10:x$.
27. $6:2=x:3$.
28. $x:8=3:4$.
29. $5:x=3:6$.
30. $3:7=5:x$.

3. If three lines are given, the fourth can be found, so as to complete a proportion by the following method.

Suppose the three lines to be a, b, and c; denote by x the fourth line, which will complete the proportion $a : b = c : x$.

From any point P draw two lines PL and PM at any angle with each other.



Beginning at P, mark off on one of the lines the distances PY equal to a, and PZ equal to b. On the other line mark off the distance PW equal to c.

Draw the line YW, and draw ZV parallel to YW.

Then the distance PV will be equal to the required line x.

That is, $$PY : PZ = PW : PV.$$

or $$a : b = c : x.$$

Also if you test the lengths of YW and ZV, you will find that they are in proportion both with PY and PZ and with PW and PV. That is,

$$YW : ZV = PY : PZ$$

$$\text{and } YW : ZV = PW : PV.$$

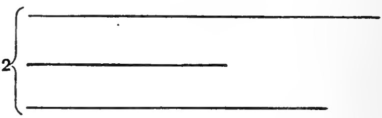
Notice also that the angles PYW and PZV are equal; also the angles PWY and PVZ.

You may find this question in proportion difficult; but you should master it, as we shall need it presently in problems about land surveying.

Surveyors use this principle constantly: they find the lengths of three lines in a proportion, and then calculate, without actually measuring, the length of a fourth line which will make the proportion complete.

Find by this method the fourth lines which will complete the following proportions:

31


32


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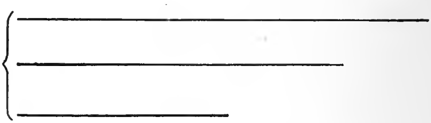

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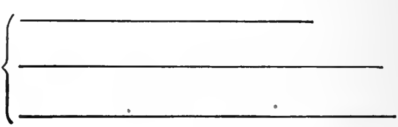
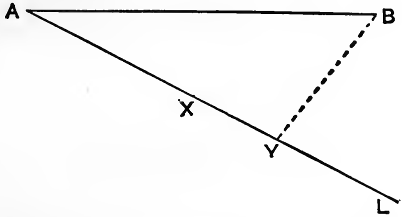


36. Mark off two-thirds of the line AB.

Hint: On any line, as AL, from A, mark off the distance AX equal to 2 units of length (centimetres, inches, etc.), and AY equal to 3 of the same units.



37. Divide the line CD into two parts, one of which shall be two-fifths of the whole line.

38. Divide the line EF into two parts, one of which shall be five-eighths of the whole line.

39. Draw a line 7 cm. long, and divide it into two parts, one of which shall be three-fifths of the whole line.

40. Draw a line 8 cm. long, and divide it into two parts, one of which shall be one-third of the whole line.

4. To divide a straight line into any given number of equal parts.

(a) With the aid of a graduated ruler.

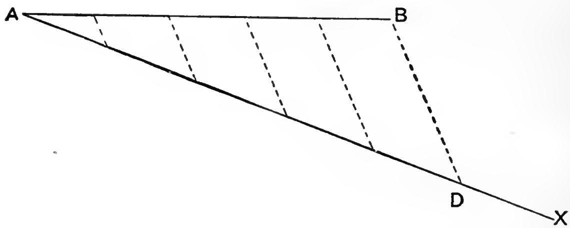
Let AB be the given straight line, and 5 the number of equal parts into which it is to be divided.

The method of division is similar to that shown on p. 26.

(b) With the aid of compasses and ungraduated ruler.

Let AB be the straight line, and 5 the number of equal parts into which it is to be divided.

From A draw AX of any convenient length and making any convenient angle with AB.



Beginning at A lay off upon AX five equal spaces of any convenient length; let D be the last point of division.

Draw a straight line from D to B, and from the other points of division on AD draw, with the aid of compasses, lines parallel to DB. These lines will divide AB into five equal parts.

(*c*) With the aid of a square or parallel rulers.

Proceed as in (*b*), but draw the parallel lines with the aid of the square or the parallel rulers.

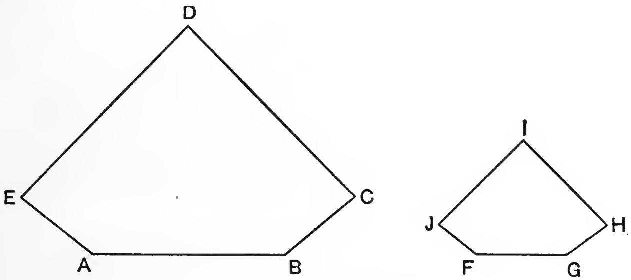
Lay off straight lines equal to the following, and divide them into equal parts as indicated: —

41	Three	equal	parts.
42	Four	"	"
43	Five	"	"
44	Three	"	"
45	Six	"	"
46	Three	"	"
47	Seven	"	"
48	Two	"	"
49	Five	"	"
50	Eight	"	"

CHAPTER XXIX

SIMILAR FIGURES

1. **Similar polygons** have the same shape; that is, one is an exact reduced copy of the other. Each angle and each side of one polygon corresponds to an angle and a side of the



other. The two angles which correspond are equal in each case; thus angle $A =$ angle F, angle $B =$ angle G, angle $C =$ angle H, etc. The equal angles lie in the same order in the two polygons; thus, if you begin at A and count around the polygon towards the right, the angles are equal, each to each, to the angles of the other polygon, beginning at F and also counted around towards the right.

The sides which correspond are *not* equal, but the lengths of any pair are in exactly the same ratio as the lengths of any other pair; thus, if AB is three times as long as FG, then BC is three times as long as GH, and CD is three times as long as HI, etc.

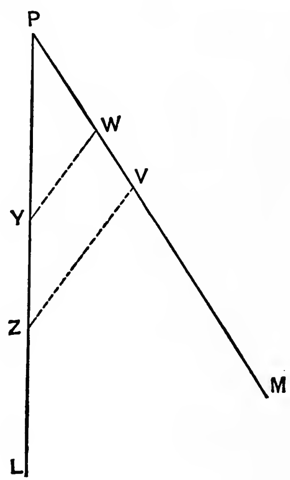
Any two pairs of corresponding sides of similar polygons form a proportion:

$$AB : FG = BC : GH$$
$$CD : HI = DE : IJ, \text{ etc.}$$

The sides which correspond have the same positions in the two polygons with respect to the equal angles, so that if you begin at A and count towards the right, the sides will correspond, each to each, to those of the other polygon, beginning at F and also counted towards the right.

Two polygons would not be similar if merely their corresponding angles were equal; for instance, a square is not similar to a rectangle. Nor would two polygons be similar if merely their corresponding sides were proportional: a square is not similar to a rhombus. Angles and sides must both be investigated before you can infer that two polygons are similar.

Triangles, however, are an exception. Two triangles whose corresponding angles are equal *must* also have their corre-

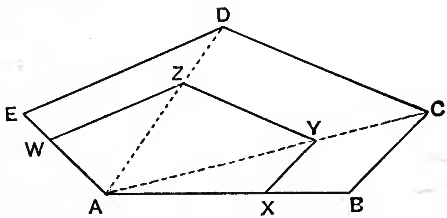


sponding sides proportional; and if you find that the sides of two triangles are proportional, you can infer that their angles are equal, each to each. You have already tested this truth in drawing proportional lines (see p. 213). In the diagram, here repeated, the similar triangles are PYW and PZV. The angles P, Y, and W correspond to P, Z, and V; $P = P$, $Y = Z$, and $W = V$. The sides PY, PW, and YW correspond to PZ, PV, and ZV; and

$$PY : PZ = PW : PV = YW : ZV.$$

2. To draw a polygon which shall be similar to a given polygon.

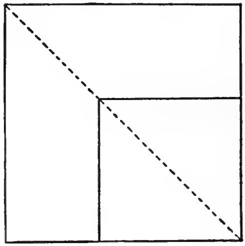
The problem of drawing a polygon which shall be similar to a given polygon is performed by dividing the given polygon into triangles and then drawing a series of triangles which shall be similar to these.



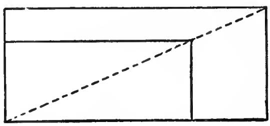
Suppose, for instance, that you wish to draw a polygon which shall be similar to $ABCDE$, but shall have sides only two-thirds as long. From any vertex, as A, draw diagonals to the other vertices, and lay off on AB the distance AX equal to two-thirds of AB. Draw XY parallel to BC; YZ parallel to CD; and ZW parallel to DE. Then the polygon $AXYZW$ will be the required polygon; for its angles are equal, each to each, to those of $ABCDE$; and its sides are each two-thirds as long as the corresponding sides of $ABCDE$.

1. Name the pairs of equal angles in the two polygons; also the pairs of corresponding sides.
2. How do the entire perimeters of the two polygons compare in length?

3. Draw a square with sides 5 cm. long, and within it another square whose sides shall be three-fifths as long.

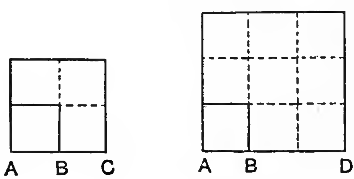


4. Draw two rectangles, one with sides 7 cm. and 3 cm. long; the other, similar to the first, but with sides five-sevenths as long.

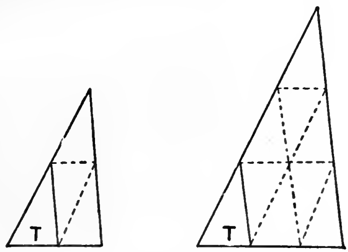


5. Draw two rhombuses, one with sides 4 cm. long, and angles of 45° and 135°; the other, similar to the first, but with sides three-fourths as long.
6. Draw two parallelograms, one with sides 6 cm. and 4 cm. long and angles 60° and 120°; the other, similar to the first, but with sides two-thirds as long.
7. Draw two triangles, one with angles 30°, 60°, and 90°, and the shortest side 3 cm. long; the other similar to the first, but with sides half as long.
8. Draw two triangles, one with a base 8 cm. long and the angles at the ends of the base 40° and 70°; the other, similar to the first, but with the corresponding base three-fourths as long.
9. Draw three parallelograms, one within the other, all similar, with angles 45° and 135°, each having sides two-thirds as long as those of the next larger, the sides of the greatest being 9 cm. and 45 mm.
10. Draw two similar pentagons, each angle of the first being 108° and each side 3 cm. long; the side of the second being two-thirds as long.

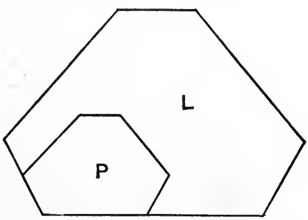
3. **Areas of Similar Polygons.** If the length of the side, AB, of a square be doubled, and another square be drawn on AC,



the new square will contain four squares, each equal to the first. If the side *AB* be tripled, and a square be drawn on *AD*, this will contain nine squares, each equal to the original one.



Likewise, if the sides of a triangle *T* be doubled, and a new triangle similar to *T* be drawn, it will contain four triangles, each equal to *T*; and tripling the sides of *T* produces a triangle whose area is nine times as great as the area of *T*.



So also with any polygons which are similar to one another, as *P* and *L*: doubling the lengths of the sides makes the area of the polygon four times as great; and so on.

This is expressed by saying that "the areas of similar polygons are to each other as the squares of any two corresponding sides."

The square of a number is the number multiplied by itself; thus the square of 5 is 25; of 7, 49; of 8, 64; of $\frac{2}{3}$, $\frac{4}{9}$; of $\frac{5}{7}$, $\frac{25}{49}$, etc.

11. If you add 3 cm. to the side of a square with an edge 2 cm. long, how much do you add to its area?

12. One side of a certain polygon is 3 cm. long, and its area is 80 sq. cm. What is the area of a similar polygon whose corresponding side is 12 cm. long?

13. Two corresponding sides of two similar polygons are 5 cm. and 7 cm. long How do their areas compare?

14. The areas of two similar polygons are 50 sq. cm. and 200 sq. cm. A certain side of the greater polygon is six inches long; what is the length of the corresponding side of the other?

15. How does the area of the diagram of the prism on p. 33 compare with the area of the diagram which is required to be drawn?

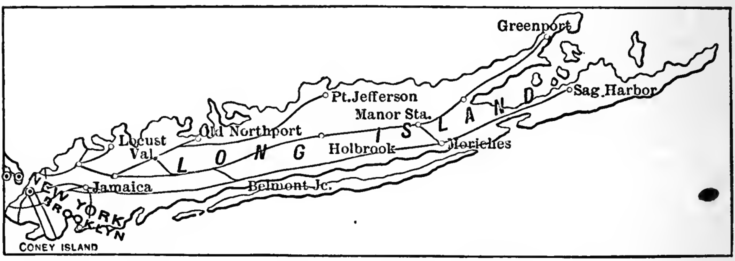
16. If you were to double the length of every line in the diagram of the parallelopiped on p. 19, by how much would you increase its area?

17. If you need paper 14 cm. × 10 cm. to draw the diagram on p. 82, as directed there, what would be the dimensions of the paper needed if the surface of the pyramid were to have only one-fourth as much area?

18. If the dodecaedron on p. 122 had an edge 1 cm. long, the area of its surface would be about 20.65 sq. cm. What would be the area if the edge were 3 cm. long?

19. The area of the State of Kentucky is about 40,000 sq. miles. What is the area of a map of Kentucky drawn on the scale of 1 : 200,000?

20. The distance in a straight line from New York City to the end of Long Island is about 115 miles. On what scale is this map drawn?

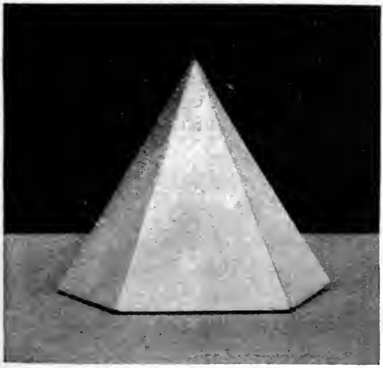


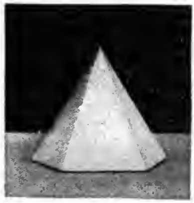
Map of Long Island

4. **Similar Polyedrons.** Two polyedrons are similar when one is an exact reduced copy of the other. In such figures the corresponding edges are proportional, the corresponding faces are similar, and the corresponding diedral angles are equal.

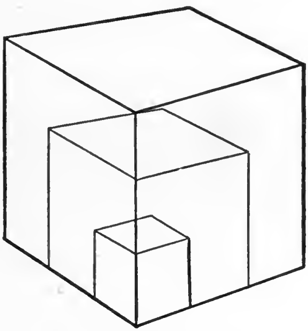
The entire surfaces are proportional to the squares of any two corresponding edges.

Let us see how their volumes compare. If the edge of a cube be doubled, and another cube be formed, it will contain eight cubes each equal to the first. If the edge of the first





cube be tripled, the new cube will contain twenty-seven cubes of the same size as the first. Thus, multiplying the edge by 2 makes the volume 8 times as great; and multiplying the edge by 3 makes the volume 27 times as great. The same would be true of any similar polyedrons, whatever were their shapes.



This is expressed by saying that "the volumes of similar polyedrons are to each other as the cubes of their corresponding edges."

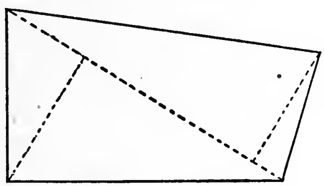
The cube of a number is the number multiplied by itself twice: thus the cube of 2 is $2 \times 2 \times 2$ or 8; the cube of 7 is $7 \times 7 \times 7$ or 343; the cube of $\frac{4}{5}$ is $\frac{4}{5} \times \frac{4}{5} \times \frac{4}{5}$ or $\frac{64}{125}$, etc.

1. What would be the effect on the volume of a cube, if its edge were lengthened so as to be five times as great as before?
2. Two corresponding edges of two similar pyramids are 3 cm. and 4 cm. long. How do their volumes compare?
3. If the octaedron shown on p. 120 had an edge 1 cm. long, its volume would be about 471 sq. mm. What is the volume of the octaedron which the directions say is to be made, which has an edge 5 cm. long?
4. If the icosaedron on p. 121 had an edge 1 cm. long, its volume would be about 2.18 cu. dcm. What is the volume of the icosaedron which is described as having an edge 2 cm. 5 mm. long?
5. If the dodecaedron on p. 122 had an edge 1 cm. long, its volume would be about 7.66 cu. cm. What is the volume of the dodecaedron to be made according to the directions, which is to have an edge about 1.9 cm. long?
6. The frustum of the pyramid described on p. 82 is the part left from a complete pyramid when a plane has been passed through, parallel to the base, and dividing the lateral edges each into two equal parts. How much of the original pyramid was removed by this plane?
7. If the diagram on p. 123 were folded up, so as to form a prism, how would its volume compare with the volume of the prism which the accompanying directions describe?
8. If Gulliver was six feet tall and his nose was two and one-fourth inches long, and one of the Liliputians was only six inches tall but shaped just like him, what was the length of the Liliputian's nose?
9. If it took 128 sq. inches of material to make Gulliver a pair of gloves, how much would it take to make a pair for the Liliputian?
10. If Gulliver weighed 180 pounds, how much did the Liliputian weigh?

CHAPTER XXX

SURVEYING

1. **Surveying.** Suppose that you were to begin at one corner of your school-yard, and measure the length of each side with the aid of a metre-stick and the size of each angle with the aid of a protractor. Then suppose that by stretching strings from corner to corner you were to divide the yard into triangles whose areas you could calculate.



Finally, suppose that you were to draw on paper a plan of the yard, setting down your measurements and calculations. In doing all this, you would be making what is called a *survey* of the yard.

To survey a piece of land is to measure its boundaries and angles, and to ascertain its shape, its area, and its situation with reference to other land. The area is found by calculations after the other measurements have been made; you have already been shown various methods.

Although each side and each angle might be measured as we have imagined you as doing in your school-yard, such a process would be tedious if the land were of considerable

size, and perhaps impossible if the measurements were interrupted by trees, houses, water, etc. So the art of a surveyor consists in making as few actual measurements as possible, and in ascertaining the rest indirectly. He does this partly by applying certain geometric principles and partly by using certain instruments.

The principles are those of similar polygons, which have already been explained on pp. 217–221, and are as follows:

I. Similar polygons have their corresponding angles equal and their corresponding sides proportional.

II. Triangles are similar in all respects,

(*a*) If their corresponding angles are equal; or

(*b*) If their corresponding sides are proportional; or

(*c*) If two corresponding sides are proportional and the angles formed by those sides are equal.

A surveyor's instruments are merely more convenient and accurate substitutes for the metre-stick and protractor.

For measuring lines he has a steel tape from 100 to 250 feet long.

For measuring angles he has a transit.

This instrument consists of a protractor, mounted on a tripod, and having a small telescope for sighting distant objects. The table of the tripod can be adjusted so as to rest exactly horizontal, and two small spirit levels on its top test whether it is in this position. The telescope is pivoted over the centre of the protractor, around which it moves accompanied by a pointer indicating the angle observed.

A plumb line is attached to the protractor at its centre, and shows the point on the ground corresponding to the vertex of the angle observed.

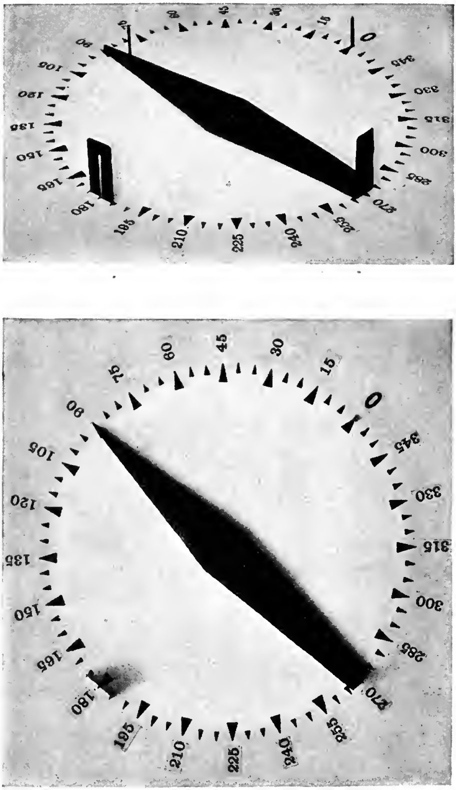
For measuring heights the transit has a second protractor which rests vertical to the first; and the telescope can be made to move on this also, accompanied by another pointer.

Surveying Instruments

Lastly, the surveyor has a levelling staff to indicate on a distant object the point which is on a horizontal level with the protractor. This is a rod about six feet long, with a sliding disk which can be adjusted so that its centre may be in the line of sight of the telescope.

If your school does not own these surveyor's instruments, there are substitutes which will serve the purpose fairly well.

For measuring lengths you can use a fifty-foot tape measure; or you can make a pole, three metres (or ten feet) long, divided into smaller parts. The pole can serve also as a levelling staff.



A Transit Board (two views)

A surveyor's transit is an expensive instrument; but a boy of some ingenuity, who has an idea of the use to which his work will be put, can make quite a serviceable substitute out of material which he is likely to have at hand. The picture of such an instrument is given here.

A protractor (360°) drawn on paper is pasted on a square board; the indicator — a small stick — turns on a screw pivot in the centre. The

ights are two nails and two bits of zinc with narrow slits, the fixed nail eing at 0° on the board. The plumb line and the tripod can be added vithout much difficulty, and then the instrument can be used with the oard on edge for taking elevations, or flat for sighting objects on a level. 3y means of the plumb line and the board on edge, the top of the tripod an be adjusted so as to be level, two positions determining the direction f the plane.

2. **Problems in Surveying.** We will suppose now that ou are provided with surveying implements — transit, tape ıeasure or measuring pole, levelling staff, and note-book — nd are ready for practical work.

Probably you will wish to take four companions, — one to old the levelling staff while you use the transit, two to meas-re base lines, and one to record the observations in a note-ook. It will be well to have every measurement repeated ıdependently by at least one member of the party; and all ıould work out the problems which result. Make your dia-rams and calculations as neat as possible.

We will consider five problems: —

. How to calculate the height of an object which stands on a horizontal plane.
. How to calculate the height of an object which you cannot approach very nearly.
. How to calculate your distance from an object without approaching it.
. How to calculate the distance between two objects without approaching either.
. How to survey a tract of land.

Every operation in surveying begins with the measurement f a *base line.* This is the distance along the ground from he plumb line under the transit to some point where the evelling staff is placed. A surveyor tries to obtain the engths of all other lines by calculation; so the measurement f the base line must be made very carefully, as an error here

will be repeated and probably increased in the rest of the work.

A *horizontal angle* between two objects is formed by two imaginary lines extending from those objects to the centre of the horizontal protractor on the transit.

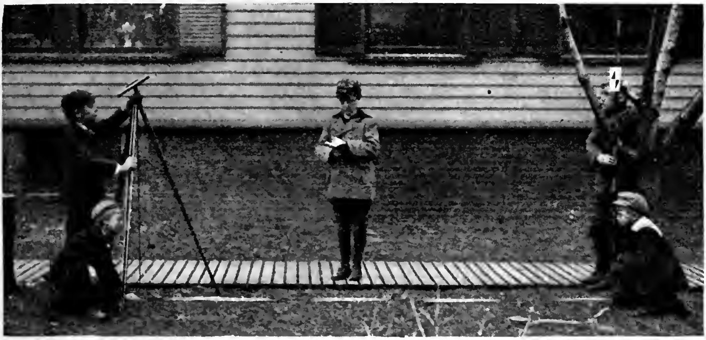
Measuring a Base Line

To measure such an angle, the surveyor after ascertaining that his transit is horizontal, sights the objects in turn, noting in each case the number of degrees indicated on the protractor by the pointer. The levelling staff is usually held at each of the objects, and the disk is raised or lowered until the centre of the disk is sighted.

An *angle of elevation* is formed by two imaginary lines extending from the top and bottom of an object to the centre of the vertical protractor on the transit. Like a horizontal angle it is measured by sighting the top and bottom of the object, and noting the degrees indicated by the pointer on the vertical protractor.

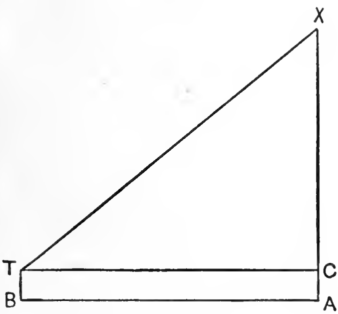
In the following problems the lower point sighted is on a horizontal level with the transit. The height of the transit, therefore, is finally added to the height of that part of the object which is obtained by calculation.

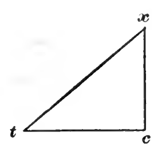
1. How to calculate the height of an object which stands on horizontal ground.



"What is the height of the tree?"

The picture shows a group of boys making measurements from which to calculate the height of a tree, represented by AX in the diagram.





The transit is placed in a convenient position (T in the diagram) with the plumb line hanging over B on the ground. The levelling staff (CA in the diagram) is placed directly under the highest point of the tree, and the disk is raised or lowered until its centre is on a horizontal line with the transit. The height CA is noted. Then the top X of the tree is sighted, and the angle CTX noted. The distance AB is measured on the ground.

These measurements are sufficient for calculating the required height AX. They should be recorded in a note-book by the boy who is acting as clerk, whose duty also is to prepare a diagram, like $ACXTB$, for future use.

The calculation should be made afterwards by each boy independently, as follows: —

Suppose the measurements taken to be,

$$\begin{aligned} CA\ (= TB) &= 4 \text{ feet} \\ AB\ (= CT) &= 25 \text{ feet} \\ \text{angle } CTX &= 39^\circ. \end{aligned}$$

Draw on paper a line ct representing CT on some convenient scale, say $\frac{1}{100}$; then, since CT is 25 feet, ct will be $\frac{1}{100}$ of 25 feet, or 3 inches long.

With the aid of a protractor, make an angle at t equal to CTX, that is, 39°, and an angle at c equal to TCX, that is, 90°, and prolong the lines until they meet at x.

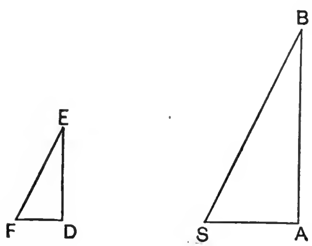
You have thus constructed a triangle ctx similar to CTX, and their corresponding sides are proportional. Measure the length of cx and compare it with the length of ct. Suppose that cx is $\frac{4}{5}$ as long as ct; then CX will be $\frac{4}{5}$ as long as CT; or, since CT is 25 feet, CX is 20 feet. To this the length of CA ($= 4$ ft.) is to be added, making the length of AX 24 feet, which is the height of the tree.

In the country or in the city you will occasionally wish to form an idea of the height of some object — a tree, flagstaff, monument, or building — without the aid of any instrument. With what you now know about similar triangles, you can do this with some accuracy, provided that the object is casting a shadow by sunlight; for close by there will probably be some smaller object — a post, for instance — also casting a shadow. You will estimate by the eye the height of the post and the length of its shadow; then, as the ratio of the taller

object to its own shadow is the same, all you will have to do is to pace off this shadow.

Measuring a Shadow

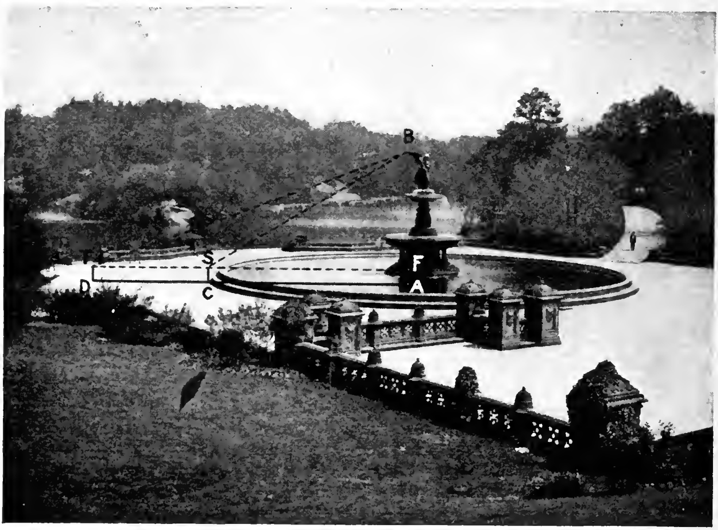
Suppose, for instance, that AB represents a tower, and AS its shadow; also that DE represents a boy standing near the tower, and DF his shadow. The triangles ABS and DEF are similar; so, if the boy is 5 ft. tall and his shadow is 4 ft.



long, the height of the tower is five-fourths of the length of its shadow. If, therefore, a boy knows that the length of his step is 21 inches, and finds that he takes 32 steps along the shadow, he has the length of the shadow 56 ft., five-fourths of which, or 70 ft., will be the height of the tower.

2. How to calculate the height of an object which you cannot approach very nearly.

Suppose AB to be an object which you cannot approach nearer than C.

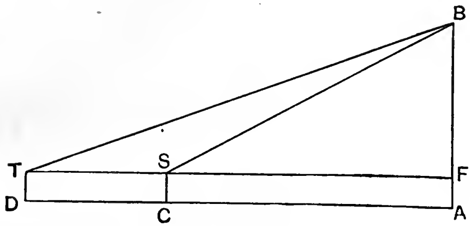


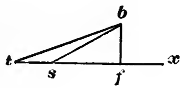
Fountain in Central Park, New York

Measure a convenient distance CD on a horizontal line with A. Set up the transit at T with the plumb line hanging over D. You will begin by sighting some point F in AB on a horizontal line with T, and with the transit measure the angle FTB. Then move the transit to S, with the plumb line hanging over C, and measure the angle FSB.

With these measurements you can calculate the height of AB.

Draw on paper a line *st* representing ST ($= CD$) on any convenient scale, and prolong the line towards x. With the





aid of a protractor make the angles *ftb* equal to *FTB*, and *fsb* equal to *FSB*. From *b* draw *bf* perpendicular to *tx*.

The triangles *STB* and *stb* are similar, and give the proportion

$$ST : SB = st : sb$$

in which *ST* and *st* are already known, and *sb* can be measured on the diagram, so that *SB* can be calculated; that is, $SB = \frac{sb \times ST}{st}$

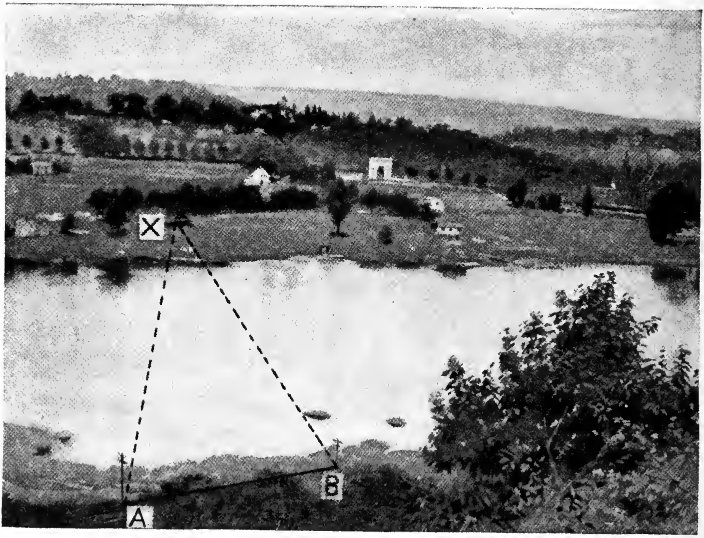
The triangles *FSB* and *fsb* are similar, and give the proportion

$$SB : FB = sb : fb$$

in which *SB* and *sb* are already known, and *fb* can be measured on the diagram, so that *FB* can be calculated; that is, $FB = \frac{SB \times fb}{sb}$

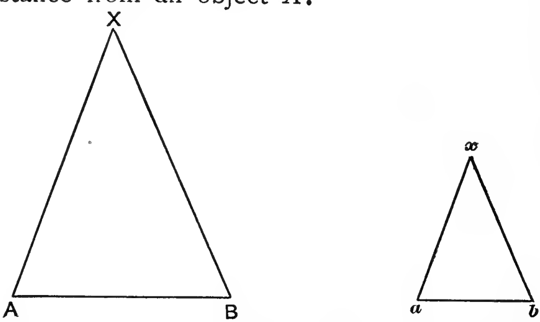
To the height of *FB* thus found you will add *DT* (= *AF*), the height of the transit, in order to obtain the total height of *AB*.

3. How to calculate your distance from an object without approaching it.



On the Harlem River

Suppose that you are at the point A and wish to know your distance from an object X.



Beginning at A measure a line AB in any convenient direction and of any convenient length. Placing the transit at A

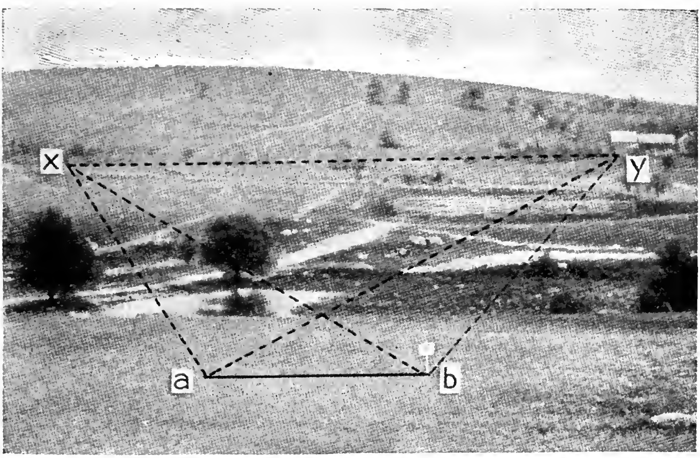
and then at B, measure the angles A and B. With these measurements you can calculate the distance AX.

Draw on paper a line ab representing AB on any convenient scale. With the aid of a protractor make the angles a equal to A, and b equal to B, forming the triangle abx. The triangles abx and ABX are similar, and give the proportion

$$ab : AB = ax : AX,$$

in which ab and AB are already known, and ax can be measured on the diagram; so that AX can be calculated, that is, $AX = \frac{AB \times ax}{a\ b}$

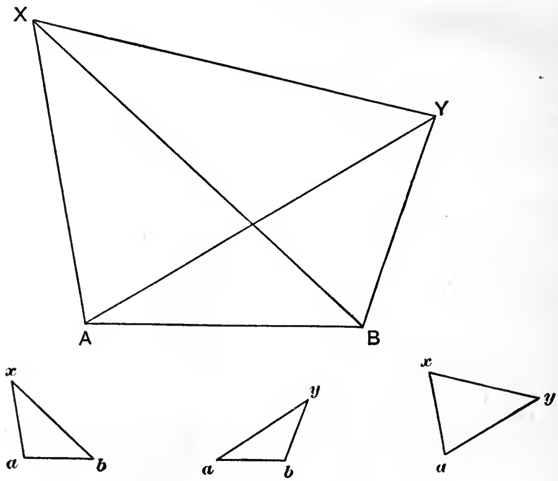
4. How to calculate the distance between two points without approaching either.



Suppose that X and Y are two points the distance between which you wish to find.

Measure a line AB in any convenient direction and of any convenient length.

Placing the transit at A, measure the angles BAX, YAX, and BAY. Then, placing the transit at B, measure the angles ABX and ABY. With these measurements you can calculate the distance XY.



First draw on paper a triangle abx, similar to ABX, ab representing AB on a reduced scale, angle bax = angle BAX, and angle abx = angle ABX. From this, by measuring ax and applying the rule of proportion, you can calculate the length of AX.

Then draw a triangle aby, similar to ABY, ab representing AB on the same reduced scale as before, angle bay = angle BAY, and angle $aby = ABY$. From this, by measuring ay and applying the rule of proportion, you can calculate the length of AY.

Lastly, draw a triangle yax similar to YAX, angle yax being equal to YAX, ax and ay having the lengths previously found. From this, by measuring the length of xy and applying the rule of proportion, you can calculate the length of XY.

5. How to survey a piece of land.

Suppose $ABCDE$ to be the land which you are to survey. To do this you will find: —

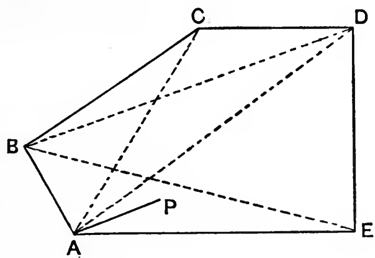
(1). The lengths of the boundaries.

(2). The direction, according to the points of the compass, in which at least one of the boundaries extends.

(3). The sizes of the angles.

(4). The area.

Lastly, you will make a plan of the land, and indicate in one corner of the paper the scale on which it is drawn.



Begin by choosing a position for your base line. This should be done carefully, so that a single base line may serve for the whole survey. One of the boundaries of the land (for instance, AB) will be your first choice; but if the boundaries are all very long, or are otherwise inconvenient to measure, you can lay off the line in some other direction, as AP.

We will suppose that AB is the base line and that it has been carefully measured. Then, using the transit, and taking the ends of the base line as vertices, measure the angles BAC, CAD, and DAE; and ABE, EBD, and DBC.

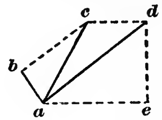
Determine the direction of AB with the aid of a compass.

With these measurements you can complete the survey by calculations.

First, by means of the problem which tells how to find

your distance from an object without approaching it, calculate the distances of A from the other corners of the land.

Next draw on paper a diagram on some convenient scale, showing the angles BAC, CAD, and DAE, and the distances AB, AC, AD, and AE.



Join the ends of these lines, forming a polygon *abcde* similar to $ABCDE$.

Measure the sides of *abcde*, and by means of a rule of proportion (see p. 212) calculate the lengths of the boundaries of the land.

Measure the angles of *abcde:* these will also be the angles of the land.

Find the area of *abcde* by any of the methods described on pp. 190–194; and by means of a rule of proportion calculate the area of the land.